On a Dirichlet problem related to anisotropic fluid flow in bidisperse porous media

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Outline

- Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain $(n = 2, 3)$ occupied by a viscous incompressible fluid.
- Let $\mathbf{u} = (u_1, \ldots, u_n)^\top$ be the velocity field and the π be the pressure field.
- Let $\mathbb{E}(\mathbf{u}) = (E_{j\beta}(\mathbf{u}))_{1 \le j,\beta \le n}$ be the strain tensor field:

$$\mathbb{E}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right), \quad E_{j\beta}(\mathbf{u}) = \frac{1}{2}\left(\partial_j u_\beta + \partial_\beta u_j\right), \quad 1 \le j, \beta \le n.$$

Introduction

The stress tensor field $\mathbb{T} = (T_{i\alpha})_{1 \le i,\alpha \le n}$ of a general (anisotropic) Newtonian fluid is given by the following constitutive relation:

$$T_{i\alpha}(\mathbf{u}, \pi) = -\pi\delta_{i\alpha} + a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), \quad 1 \le i, \alpha \le n$$

where $\mathbb{A} = \left(a_{ij}^{\alpha\beta}\right)_{1 \le i,j,\alpha,\beta \le n}$ is the viscosity tensor coefficient, whose entries satisfy[1]:

$$a_{ij}^{\alpha\beta} = a_{\alpha j}^{i\beta} = a_{i\beta}^{\alpha j} \quad 1 \le i, j, \alpha, \beta \le n.$$

[1] The symmetries of $\mathbb{A}$ are imposed by the symmetries of $\mathbb{T}$ and $\mathbb{E}(\mathbf{u})$.

The divergence of the stress tensor field can be written component-wise in the following manner:

$$(\text{div } \mathbb{T})_i = \partial_\alpha T_{i\alpha} = \partial_\alpha \left(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}) \right) - \delta_{i\alpha}\partial_\alpha \pi, \quad i = \overline{1, n}. \tag{1}$$

We introduce the following second-order differential operator given in component-wise divergence form as[2]:

$$(\mathcal{L}\mathbf{u})_i = \partial_\alpha \left(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}) \right), \quad i = \overline{1, n}.$$

Then:

$$\text{div } \mathbb{T} = \mathcal{L}\mathbf{u} - \nabla\pi$$

In particular, in the **isotropic case**, the entries of $\mathbb{A}$ are given by:

$$a_{ij}^{\alpha\beta} = \mu(\delta_{\alpha j}\delta_{\beta i} + \delta_{\alpha\beta}\delta_{ij}), \quad 1 \leq i, j, \alpha, \beta \leq n$$

Furthermore, for constant μ, the operator $\mathcal{L}$ has the expression $\mathcal{L}\mathbf{u} = \mu\Delta\mathbf{u}$.

[2] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "Dirichlet and transmission problems for anisotropic Stokes and Navier-Stokes systems with L_∞ tensor coefficient under relaxed ellipticity condition". 2021.

The equations:

$$\boxed{\mathcal{L}\mathbf{u} - \nabla\pi - \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}, \quad \mathrm{div}\ \mathbf{u} = 0 \qquad \text{in } \Omega} \tag{2}$$

determine the (steady-state) **anisotropic Navier–Stokes** system.

When $\kappa = 0$, the system (2) reduces to the **anisotropic Stokes system**:

$$\boxed{\mathcal{L}\mathbf{u} - \nabla\pi = \mathbf{f}, \quad \mathrm{div}\ \mathbf{u} = 0 \qquad \text{in } \Omega} \tag{3}$$

Boundary value problems for the anisotropic Stokes and Navier–Stokes systems were studied by Kohr, Mikhailov, and Wendland[3,4,5] using variational and fixed-point techniques.

[3] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "Dirichlet and transmission problems for anisotropic Stokes and Navier-Stokes systems with L_∞ tensor coefficient under relaxed ellipticity condition". 2021.

[4] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "On some mixed-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

[5] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "Non-homogeneous Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

The Navier–Stokes system (2) can be generalized in order to describe **flows in porous media**[6]. In the anisotropic case, such a model is given by the system:

$$\boxed{\mathcal{L}\mathbf{u} - \nabla\pi - \eta\mathbf{u} - \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda|\mathbf{u}|\mathbf{u} = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega} \tag{4}$$

where η, κ, λ are parameters which depend on the physical properties of the fluid and the porous medium.

We refer to (4) as the (nonlinear) **anisotropic Darcy–Forchheimer–Brinkman** system. By following similar techniques as in Kohr et al.[7,8], we study the Dirichlet problem for such a system.

[6] D. A. Nield and A. Bejan. *Convection in Porous Media*. 2013.

[7] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "Non-homogeneous Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

[8] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "On some mixed-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

A **bidisperse porous medium** is a material composed of clusters of large particles that are agglomerations of small particles, or, alternatively, a standard porous medium in which fractures or tunnels have been introduced[9].

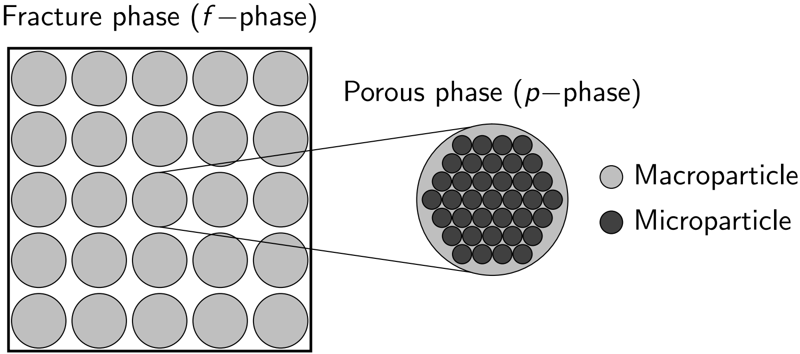

Figure: Cavity filled with a bidisperse porous medium

[9] D. Nield and A. Kuznetsov. "The onset of convection in a bidisperse porous medium". 2006.

Nield and Kuznetsov[10,11] considered the following two-velocity model for steady-state momentum transfer in a **bidisperse porous medium** by extending the Brinkman model from the monodisperse case (* denotes dimensional variables):

$$\begin{cases} \mathbf{G} = \left(\frac{\mu}{K_f}\right)\mathbf{v}_f^* + \zeta(\mathbf{v}_f^* - \mathbf{v}_p^*) - \tilde{\mu}_f(\nabla^*)^2\mathbf{v}_f^* + \frac{c_f\rho}{K_f^{1/2}}|\mathbf{v}_f^*|\mathbf{v}_f^* \\ \mathbf{G} = \left(\frac{\mu}{K_p}\right)\mathbf{v}_p^* + \zeta(\mathbf{v}_p^* - \mathbf{v}_f^*) - \tilde{\mu}_p(\nabla^*)^2\mathbf{v}_p^* + \frac{c_p\rho}{K_p^{1/2}}|\mathbf{v}_p^*|\mathbf{v}_p^* \end{cases}$$

where:

- $\mathbf{G}$ is the negative of the pressure gradient
- μ, ρ are the fluid viscosity and density
- $\tilde{\mu}_{f,p}$ and $K_{f,p}$ are the effective viscosities and permeabilities for the two phases
- $\mathbf{v}_f^*, \mathbf{v}_p^*$ are velocity fields associated with the two phases
- ζ is the coefficient for momentum transfer between the two phases

[10] D. Nield and A. Kuznetsov. "The onset of convection in a bidisperse porous medium". 2006.

[11] D. A. Nield and A. V. Kuznetsov. "A Note on Modeling High Speed Flow in a Bidisperse Porous Medium". 2012.

Kohr and Precup[12] studied the homogeneous Dirichlet problem for the following system of coupled Navier–Stokes-type equations:

$$\begin{cases} -\mu_1 \Delta \mathbf{u}_1 + \eta_1 \mathbf{u}_1 + \kappa_1 (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_1 + \nabla p_1 = \mathbf{h}_1 - \alpha_1 |\mathbf{u}_1|^{p-1} \mathbf{u}_1 - \gamma_1 (\mathbf{u}_1 - \mathbf{u}_2) & \text{in } \Omega \\ -\mu_2 \Delta \mathbf{u}_2 + \eta_2 \mathbf{u}_2 + \kappa_2 (\mathbf{u}_2 \cdot \nabla) \mathbf{u}_2 + \nabla p_2 = \mathbf{h}_2 - \alpha_2 |\mathbf{u}_2|^{p-1} \mathbf{u}_2 - \gamma_2 (\mathbf{u}_2 - \mathbf{u}_1) & \text{in } \Omega \\ \operatorname{div} \mathbf{u}_1 = 0, \quad \operatorname{div} \mathbf{u}_2 = 0 & \text{in } \Omega \end{cases} \tag{5}$$

with constant parameters $\mu_i, \gamma_i > 0, \eta_i, \kappa_i, \alpha_i \geq 0, i = 1, 2, p \geq 1$.

By following a similar approach, we study the **non-homogeneous** Dirichlet problem for a system of two **coupled anisotropic Darcy–Forchheimer–Brinkman** equations.

[12] M. Kohr and R. Precup. "Analysis of Navier-Stokes Models for Flows in Bidisperse Porous Media". 2023.

Dirichlet problem for the anisotropic D–F–B system

The linear homogeneous case

We consider the case $\kappa = \lambda = 0$ and study the homogeneous Dirichlet problem:

$$\begin{cases} \mathcal{L}\mathbf{u} - \nabla\pi - \eta\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \operatorname{div}\, \mathbf{u} = 0 & \text{in } \Omega \\ \operatorname{Tr}\, \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where:

- $\eta \in L^\infty(\Omega)$ is a parameter which satisfies the positivity condition:

$$\langle \eta\mathbf{w}, \mathbf{w}\rangle_\Omega \geq 0, \quad \forall \mathbf{w} \in H^1(\Omega)^n$$

- $\mathbf{f} \in H^{-1}(\Omega)^n$ is a given distribution
- $\operatorname{Tr} : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is the trace operator

We assume that:

(A1) $a_{ij}^{\alpha\beta} \in L^\infty(\Omega)$ for any $1 \leq i, j, \alpha, \beta \leq n$ and denote:

$$\|\mathbb{A}\| = \max_{1 \leq i,j,\alpha,\beta \leq n} \left\| a_{ij}^{\alpha\beta} \right\|_{L^\infty(\Omega)}.$$

(A2) $\mathbb{A}$ satisfies the **ellipticity condition** only in terms all **symmetric matrices** in $\mathbb{R}^{n\times n}$ with **zero trace**, i.e.,

$$\begin{aligned} \exists C_{\mathbb{A}} > 0 \text{ s.t. } a_{ij}^{\alpha\beta} \xi_{i\alpha} \xi_{j\beta} \geq C_{\mathbb{A}}^{-1} |\boldsymbol{\xi}|^2, \quad \forall \boldsymbol{\xi} = (\xi_{i\alpha})_{1 \leq i,\alpha \leq n} \in \mathbb{R}^{n\times n} \\ \text{s.t. } \boldsymbol{\xi} = \boldsymbol{\xi}^\top \text{ and } \sum_{i=1}^{n} \xi_{ii} = 0, \end{aligned} \tag{7}$$

where $|\boldsymbol{\xi}|^2 = \xi_{i\alpha}\xi_{i\alpha}$.

We introduce the bilinear forms:

$$a_{\mathbb{A},\eta;\Omega} : \mathring{H}^1(\Omega)^n \times \mathring{H}^1(\Omega)^n \to \mathbb{R}, \quad b_\Omega : \mathring{H}^1(\Omega)^n \times L^2(\Omega)/\mathbb{R} \to \mathbb{R}$$

given by:

$$\begin{aligned} a_{\mathbb{A},\eta;\Omega}(\mathbf{u},\mathbf{v}) &= \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v}) \right\rangle_\Omega + \langle \eta \mathbf{u}, \mathbf{v} \rangle_\Omega \\ b_\Omega(\mathbf{v}, q) &= -\langle \operatorname{div} \mathbf{v}, q \rangle_\Omega \end{aligned}, \quad \begin{aligned} &\forall \mathbf{u}, \mathbf{v} \in \mathring{H}^1(\Omega)^n \\ &\forall q \in L^2(\Omega)/\mathbb{R} \end{aligned} \tag{8}$$

and state the following[13]

Lemma

The Dirichlet problem (6) is equivalent to the following variational problem:

$$\begin{cases} a_{\mathbb{A},\eta;\Omega}(\mathbf{u},\mathbf{v}) + b_\Omega(\mathbf{v},\pi) = -\langle \mathbf{f}, \mathbf{v} \rangle_\Omega \\ b_\Omega(\mathbf{u}, q) = 0 \end{cases}, \quad \begin{aligned} &\forall \mathbf{v} \in \mathring{H}^1(\Omega)^n \\ &\forall q \in L^2(\Omega)/\mathbb{R} \end{aligned}. \tag{9}$$

[13] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "Dirichlet and transmission problems for anisotropic Stokes and Navier-Stokes systems with L_∞ tensor coefficient under relaxed ellipticity condition". 2021.

The primary result used in the analysis of problem (9) is the following theorem[14]:

Theorem

Let X and M be two Hilbert spaces and let $a : X \times X \to \mathbb{R}$, $b : X \times M \to \mathbb{R}$ be two **bounded bilinear forms**. *Let $f \in X'$ and $g \in M'$. Denote by V the subspace of X defined as:*

$$V = \{v \in X \mid b(v, q) = 0, \ \forall q \in M\}$$

Assume that:

1. *The bilinear form a is* **coercive**, *i.e., there exists a constant $c_a > 0$ such that:*

$$a(u, u) \geq c_a \|u\|_X^2, \quad \forall u \in X.$$

2. *The bilinear form b satisfies the* **inf-sup** *condition:*

$$\inf_{q \in M \setminus \{0\}} \sup_{v \in X \setminus \{0\}} \frac{b(v, q)}{\|v\|_X \|q\|_M} \geq c_b$$

for some constant $c_b > 0$.

[14] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*. 2004.

Theorem (cont.)

Then, for unknowns $(u, p) \in X \times M$*, the mixed variational problem:*

$$\begin{cases} a(u, v) + b(v, p) = f(v), & \forall v \in V \\ b(u, q) = g(q), & \forall q \in M \end{cases} \tag{10}$$

is **well-posed***, i.e., it has a unique solution* $(u, p) \in X \times M$ *and there exists a constant* $C > 0$ *such that:*

$$\|u\|_X + \|p\|_M \leq C \left(\|f\|_{X'} + \|g\|_{M'} \right).$$

Denote by $\mathring{H}^1_{\mathrm{div}}(\Omega)^n$ the space:

$$\mathring{H}^1_{\mathrm{div}}(\Omega)^n = \{\mathbf{w} \in \mathring{H}^1(\Omega)^n \mid \mathrm{div}\ \mathbf{w} = 0\}.$$

Indeed, the bilinear forms $a_{\mathbb{A},\eta;\Omega}$ and b_Ω satisfy the requirements of the previous theorem[14].

Lemma

- *The bilinear form* $a_{\mathbb{A};\Omega} : \mathring{H}^1(\Omega)^n \times \mathring{H}^1(\Omega)^n \to \mathbb{R}$ *is* **bounded**, *i.e., there exists a constant* $c > 0$ *such that:*

$$|a_{\mathbb{A},\eta;\Omega}(\mathbf{u},\mathbf{v})| \leq c\|\mathbf{u}\|_{H^1(\Omega)^n}\|\mathbf{v}\|_{H^1(\Omega)^n}, \quad \forall \mathbf{u},\mathbf{v} \in \mathring{H}^1(\Omega)^n.$$

- *The bilinear form* $a_{\mathbb{A},\eta;\Omega} : \mathring{H}^1_{\mathrm{div}}(\Omega)^n \times \mathring{H}^1_{\mathrm{div}}(\Omega)^n \to \mathbb{R}$ *is* **coercive**, *i.e., there exists a constant* $c_a > 0$ *such that:*

$$a_{\mathbb{A},\eta;\Omega}(\mathbf{u},\mathbf{u}) \geq c_a\|\mathbf{u}\|^2_{H^1(\Omega)^n}, \quad \forall \mathbf{u} \in \mathring{H}^1_{\mathrm{div}}(\Omega)^n.$$

- *The bilinear form* b_Ω *is* **bounded** *and satisfies the* **inf-sup** *condition:*

$$\inf_{q \in L^2(\Omega)/\mathbb{R}\setminus\{0\}} \sup_{\mathbf{v} \in \mathring{H}^1(\Omega)^n\setminus\{0\}} \frac{b_\Omega(\mathbf{v},q)}{\|\mathbf{v}\|_{\mathring{H}^1(\Omega)^n}\|q\|_{L^2(\Omega)/\mathbb{R}}} \geq c_b,$$

for some constant $c_b > 0$.

[14] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "Dirichlet and transmission problems for anisotropic Stokes and Navier-Stokes systems with L_∞ tensor coefficient under relaxed ellipticity condition". 2021.

In view of the equivalence between the homogeneous Dirichlet problem (6) and the mixed variational problem (9) we state the following

Theorem

For any given $\mathbf{f} \in H^{-1}(\Omega)^n$ *the homogeneous Dirichlet problem* (6) *has a unique solution* $(\mathbf{u}, \pi) \in \mathring{H}^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ *and there exists a constant* $C > 0$ *such that:*

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)^n}.$$

The previous result can be extended to the non-homogeneous Dirichlet problem (following a similar argument as in Kohr et al. 2022[15]):

$$\begin{cases} \mathcal{L}\mathbf{u} - \nabla\pi - \eta\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathrm{div}\ \mathbf{u} = 0 & \text{in } \Omega \\ \mathrm{Tr}\ \mathbf{u} = \varphi & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Theorem

For any given $(\mathbf{f}, \varphi) \in H^{-1}(\Omega)^n \times H_\nu^{\frac{1}{2}}(\partial\Omega)^n$, *the Dirichlet problem* (11) *has a unique solution* $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ *and there exists a constant* $C > 0$ *such that:*

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\left(\|\mathbf{f}\|_{H^{-1}(\Omega)^n} + \|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\right).$$

[15] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "Non-homogeneous Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

Remark

The solution of the Dirichlet problem (11) *can be represented as* $(\mathbf{u}, \pi) = \mathcal{B}(\mathbf{f}, \varphi)$, *where:*

$$\mathcal{B} : H^{-1}(\Omega)^n \times H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^n \to H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R},$$

$\mathcal{B} = \mathcal{B}_{\Omega,\mathbb{A},\eta}$ *is a linear and bounded operator.*

We now consider the Dirichlet problem for the D–F–B system in the general case:

$$\begin{cases} \mathcal{L}\mathbf{u} - \nabla\pi - \eta\mathbf{u} - \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda|\mathbf{u}|\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathrm{div}\ \mathbf{u} = 0 & \text{in } \Omega \\ \mathrm{Tr}\ \mathbf{u} = \boldsymbol{\varphi} & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $\eta, \kappa, \lambda \in L^\infty(\Omega)$ are parameters and

$$\mathbf{f} \in H^{-1}(\Omega)^n, \quad \boldsymbol{\varphi} \in H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^n$$

are the given data.

To obtain norm estimates for the nonlinear terms in (12) we apply the **Hölder inequality** along with the following embedding result (particularization of the **Sobolev embedding theorem** for $n = 2, 3$):

Theorem

For any $r \in [2, 6]$ we have the embedding $H^1(\Omega)^n \hookrightarrow L^r(\Omega)$ and there exists a constant $C_r > 0$ such that:

$$\|\mathbf{u}\|_{L^r(\Omega)^n} \leq C_r \|\mathbf{u}\|_{H^1(\Omega)^n}, \quad \forall \mathbf{u} \in H^1(\Omega)^n. \tag{13}$$

The following inequalities hold[16,17]:

Lemma

Let $\mathbf{u}, \mathbf{v} \in H^1(\Omega)^n$. *Then:*

$$\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|_{H^{-1}(\Omega)^n} \le C_4^2 \|\mathbf{u}\|_{H^1(\Omega)^n}\|\mathbf{v}\|_{H^1(\Omega)^n},$$

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{v}\cdot\nabla)\mathbf{v}\|_{H^{-1}(\Omega)^n} \le C_4^2 \left(\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\mathbf{v}\|_{H^1(\Omega)^n}\right)\|\mathbf{u}-\mathbf{v}\|_{H^1(\Omega)^n}.$$

Also:

$$\||\mathbf{u}|\mathbf{v}\|_{H^{-1}(\Omega)^n} \le C_2 C_4^2 \|\mathbf{u}\|_{H^1(\Omega)^n}\|\mathbf{v}\|_{H^1(\Omega)^n},$$

$$\|(|\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v})\|_{H^{-1}(\Omega)^n} \le C_2 C_4^2 \left(\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\mathbf{v}\|_{H^1(\Omega)^n}\right)\|\mathbf{u}-\mathbf{v}\|_{H^1(\Omega)^n}.$$

[16] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "Dirichlet and transmission problems for anisotropic Stokes and Navier-Stokes systems with L_∞ tensor coefficient under relaxed ellipticity condition". 2021.

[17] M. Kohr, S. E. Mikhailov, and W. L. Wendland. "On some mixed-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

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The nonlinear case: Fixed-point formulation

Let $\mathbf{u} \in H^1_{\mathrm{div}}(\Omega)^n$. We have previously shown that the Dirichlet problem:

$$\begin{cases} \mathcal{L}\mathbf{v} - \nabla p - \eta\mathbf{v} = \mathbf{f} + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u} & \text{in } \Omega \\ \mathrm{div}\ \mathbf{v} = 0 & \text{in } \Omega \\ \mathrm{Tr}\ \mathbf{v} = \boldsymbol{\varphi} & \text{on } \partial\Omega \end{cases} \tag{14}$$

has a unique solution $(\mathbf{v}, p) \in H^1_{\mathrm{div}}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$. Let

$$(\boldsymbol{\mathcal{U}}, \mathcal{P}) : H^1_{\mathrm{div}}(\Omega)^n \to H^1_{\mathrm{div}}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$$

be operators that map $\mathbf{u}$ to the unique solution $(\mathbf{v}, p)$ of (14), i.e.,

$$(\boldsymbol{\mathcal{U}}(\mathbf{u}), \mathcal{P}(\mathbf{u})) = (\mathbf{v}, p) = \mathcal{B}\left(\mathbf{f} + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u}, \boldsymbol{\varphi}\right).$$

Then, (12) can be re-formulated as a fixed point problem:

$$\begin{cases} \mathcal{L}\boldsymbol{\mathcal{U}}(\mathbf{u}) - \nabla\mathcal{P}(\mathbf{u}) - \eta\boldsymbol{\mathcal{U}}(\mathbf{u}) = \mathbf{f} + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u} & \text{in } \Omega \\ \mathrm{div}\ \boldsymbol{\mathcal{U}}(\mathbf{u}) = 0 & \text{in } \Omega \\ \mathrm{Tr}\ \boldsymbol{\mathcal{U}}(\mathbf{u}) = \boldsymbol{\varphi} & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Since the operator $\mathcal{B}$ is bounded, there exists a constant $C_* > 0$ such that:

$$\begin{aligned}\|(\mathcal{U}(\mathbf{u}),\mathcal{P}(\mathbf{u}))\|_{H^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}} &= \|\mathcal{B}\left(\mathbf{f}+\kappa(\mathbf{u}\cdot\nabla)\mathbf{u}+\lambda|\mathbf{u}|\mathbf{u},\varphi\right)\|_{H^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}}\\ &\leq C_*\|(\mathbf{f}+\kappa(\mathbf{u}\cdot\nabla)\mathbf{u}+\lambda|\mathbf{u}|\mathbf{u},\varphi)\|_{H^{-1}(\Omega)^n\times H^{\frac{1}{2}}(\partial\Omega)^n},\end{aligned}$$

hence:

$$\|(\mathcal{U}(\mathbf{u}),\mathcal{P}(\mathbf{u}))\|_{H^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}} \leq C_*\left(\|(\mathbf{f},\varphi)\|_{H^{-1}(\Omega)^n\times H^{\frac{1}{2}}(\partial\Omega)^n}+\|\kappa(\mathbf{u}\cdot\nabla)\mathbf{u}+\lambda|\mathbf{u}|\mathbf{u}\|_{H^{-1}(\Omega)^n}\right).$$

In view of the previously stated estimates, we have:

$$\|(\mathcal{U}(\mathbf{u}),\mathcal{P}(\mathbf{u}))\|_{H^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}} \leq C_*\|(\mathbf{f},\varphi)\|_{H^{-1}(\Omega)^n\times H^{\frac{1}{2}}(\partial\Omega)^n}+CC_*\|\mathbf{u}\|^2_{H^1(\Omega)^n}, \qquad (16)$$

where $C = C_4^2\|\kappa\|_{L^\infty(\Omega)} + C_2C_4^2\|\lambda\|_{L^\infty(\Omega)}$.

Let $L \in (0,1)$ be fixed and take:

$$r = \frac{L}{2CC_*}, \quad R = \frac{r - r^2 CC_*}{C_*} > 0.$$

Theorem

Assuming that

$$\|(\mathbf{f}, \varphi)\|_{H^{-1}(\Omega)^n \times H^{\frac{1}{2}}(\partial\Omega)^n} \leq R, \tag{17}$$

the Dirichlet problem (12) *has a unique solution* $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ *with* $\|\mathbf{u}\|_{H^1(\Omega)^n} \leq r$ *and there exists a constant* $\tilde{C} > 0$ *such that:*

$$\|(\mathbf{u}, \pi)\|_{H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}} \leq \tilde{C}\|(\mathbf{f}, \varphi)\|_{H^{-1}(\Omega)^n \times H^{\frac{1}{2}}(\partial\Omega)^n}.$$

Proof

Let $\mathbf{B}_r$ be the zero-centered closed ball in $H^1_{\mathrm{div}}(\Omega)^n$ of radius r:

$$\mathbf{B}_r = \{\mathbf{v} \in H^1_{\mathrm{div}}(\Omega)^n \mid \|\mathbf{v}\|_{H^1(\Omega)^n} \leq r\}.$$

We will prove that $\boldsymbol{\mathcal{U}}$ has a fixed point in $\mathbf{B}_r$. First, note that, according to (16) and (17):

$$\|(\boldsymbol{\mathcal{U}}(\mathbf{u}), \mathcal{P}(\mathbf{u}))\|_{H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}} \leq r, \quad \forall \mathbf{u} \in \mathbf{B}_r,$$

which means that $\boldsymbol{\mathcal{U}} : \mathbf{B}_r \to \mathbf{B}_r$ is well-defined, i.e., the operator $\boldsymbol{\mathcal{U}}$ maps the ball $\mathbf{B}_r$ to itself. Second, for any $\mathbf{u}, \mathbf{v} \in \mathbf{B}_r$, in view of the previously mentioned norm estimates, we have:

$$\begin{aligned}
\|\boldsymbol{\mathcal{U}}(\mathbf{u}) - \boldsymbol{\mathcal{U}}(\mathbf{v})\|_{H^1(\Omega)^n} &\leq C_* \|\kappa\left((\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{v}\cdot\nabla)\mathbf{v}\right) + \lambda\left(|\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v}\right)\|_{H^{-1}(\Omega)^n} \\
&\leq C_* \left(C_4^2\|\kappa\| + C_2 C_4^2 \|\lambda\|\right)\left(\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\mathbf{v}\|_{H^1(\Omega)^n}\right)\|\mathbf{u}-\mathbf{v}\|_{H^1(\Omega)^n} \\
&\leq 2rCC_*\|\mathbf{u}-\mathbf{v}\|_{H^1(\Omega)^n} = L\|\mathbf{u}-\mathbf{v}\|_{H^1(\Omega)^n}.
\end{aligned} \tag{18}$$

Since $L \in (0,1)$, we obtain that $\boldsymbol{\mathcal{U}} : \mathbf{B}_r \to \mathbf{B}_r$ is a contraction. Applying the **Banach contraction principle**, we conclude that there exists a unique fixed point $\mathbf{u}^* \in \mathbf{B}_r$ of $\boldsymbol{\mathcal{U}}$, which yields a solution of problem (12).

□

Dirichlet problem for coupled anisotropic D–F–B equations

We consider the Dirichlet problem for a system of two **coupled anisotropic Darcy–Forchheimer–Brinkman equations**:

$$\begin{cases} \mathcal{L}_1\mathbf{u}_1 - \nabla\pi_1 - \eta_1\mathbf{u}_1 - \kappa_1(\mathbf{u}_1\cdot\nabla)\mathbf{u}_1 - \lambda_1|\mathbf{u}_1|\mathbf{u}_1 = \mathbf{f}_1 + \mathbf{F}_1(\mathbf{u}_1,\mathbf{u}_2) & \text{in } \Omega \\ \mathcal{L}_2\mathbf{u}_2 - \nabla\pi_2 - \eta_2\mathbf{u}_2 - \kappa_2(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2 - \lambda_2|\mathbf{u}_2|\mathbf{u}_2 = \mathbf{f}_2 + \mathbf{F}_2(\mathbf{u}_1,\mathbf{u}_2) & \text{in } \Omega \\ \mathrm{div}\ \mathbf{u}_1 = 0, \quad \mathrm{div}\ \mathbf{u}_2 = 0 & \text{in } \Omega \\ \mathrm{Tr}\ \mathbf{u}_1 = \boldsymbol{\varphi}_1, \quad \mathrm{Tr}\ \mathbf{u}_2 = \boldsymbol{\varphi}_2 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where

$$\eta_i, \kappa_i, \lambda_i \in L^\infty(\Omega), \quad i = \overline{1,2}$$

are parameters, and

$$\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in H^{-1}(\Omega)^{2n}, \quad \boldsymbol{\varphi} = (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \in H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^{2n}$$

are the given data. The equations are coupled through the operator

$$\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2), \quad \mathbf{F}_i : H^1(\Omega)^{2n} \to H^{-1}(\Omega)^n, \quad i = \overline{1,2}. \tag{20}$$

Let (X, d) be a metric space. We introduce the following **vector-valued metric** on the product space $X^2 = X \times X$ (see, e.g., Precup[18]):

$$D : X^2 \to \mathbb{R}^2, \quad D(x, y) = \begin{bmatrix} d(x_1, y_1) \\ d(x_2, y_2) \end{bmatrix}, \quad \forall x = (x_1, x_2), y = (y_1, y_2) \in X^2.$$

We refer to the pair (X^2, D) as a **generalized metric space**.

An operator $F : X^2 \to X^2$ is said to be **Lipschitz** if there exists a square matrix $M \in \mathbb{R}^{2\times 2}$ with non-negative entries such that:

$$D(F(x), F(y)) \leq MD(x, y), \quad \forall x, y \in X^2.$$

In particular, if the matrix M **converges to zero**, i.e., $\lim_{p\to\infty} M^p = 0$, the operator F is called a **generalized contraction**.

[18] R. Precup. "The role of matrices that are convergent to zero in the study of semilinear operator systems". 2009.

Regarding the "convergence to zero" property of a square matrix $M \in \mathbb{R}^{2\times 2}$, we state the following characterization result[18]:

Lemma

The following statements are equivalent:

- *The matrix M converges to zero.*
- *The matrix $I_2 - M$ is invertible and $(I_2 - M)^{-1}$ has non-negative entries.*
- *For any $\lambda \in \mathbb{C}$ such that $\det(M - \lambda I_2) = 0$ we have $|\lambda| < 1$.*

[18] R. Precup. "The role of matrices that are convergent to zero in the study of semilinear operator systems". 2009.

The following result, referred to as the **Perov fixed point theorem**[18,19], provides a generalization of the Banach contraction principle.

Theorem (Perov, 1966)

Assume that (X^2, D) is a complete generalized metric space and let $F : X^2 \to X^2$ be a generalized contraction. Then, there exists a unique $u^* \in X^2$ such that $F(u^*) = u^*$.

In particular, if $(X, \|\cdot\|)$ is a normed space, we introduce a **vector-valued norm** on the product space $X^2 = X \times X$ in the following manner:

$$|||\cdot||| : X^2 \to \mathbb{R}^2, \quad |||x||| = \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix}, \quad \forall x = (x_1, x_2) \in X^2.$$

[18] R. Precup. *Methods in Nonlinear Integral Equations*. 2002.

[19] R. Precup. "The role of matrices that are convergent to zero in the study of semilinear operator systems". 2009.

Regarding the operator $\mathbf{F}$, we assume that[20]:

(A3) The components $\mathbf{F}_1, \mathbf{F}_2$ of $\mathbf{F}$ satisfy:

$$\mathbf{F}_i(0) = 0, \quad i = \overline{1,2}.$$

(A4) The following Lipschitz condition holds for the operator $\mathbf{F}$:

$$|||\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v})|||_{H^{-1}(\Omega)^{2n}} \le A|||\mathbf{u} - \mathbf{v}|||_{H^1(\Omega)^{2n}}, \quad \forall \mathbf{u}, \mathbf{v} \in H^1(\Omega)^{2n} \tag{21}$$

for some matrix $A = (a_{ij})_{1 \le i,j \le 2} \in \mathbb{R}^{2\times 2}$ with non-negative entries.

Remark

For describing flows in **bidisperse porous media**, *we will consider* $\mathbf{F}$ *of the form:*

$$\mathbf{F}_1(\mathbf{u}_1, \mathbf{u}_2) = \gamma_1(\mathbf{u}_1 - \mathbf{u}_2), \quad \mathbf{F}_2(\mathbf{u}_1, \mathbf{u}_2) = \gamma_2(\mathbf{u}_2 - \mathbf{u}_1), \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in H^1_{\mathrm{div}}(\Omega)^n,$$

with non-negative parameters $\gamma_1, \gamma_2 \in L^\infty(\Omega)$.

Immediately, from (A3) and (A4) we deduce:

$$|||\mathbf{F}(\mathbf{u})|||_{H^{-1}(\Omega)^{2n}} \le A|||\mathbf{u}|||_{H^1(\Omega)^{2n}}, \quad \forall \mathbf{u} \in H^1(\Omega)^{2n}. \tag{22}$$

[20] M. Kohr and R. Precup. "Analysis of Navier-Stokes Models for Flows in Bidisperse Porous Media". 2023.

Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in H^1_{\mathrm{div}}(\Omega)^{2n}$ be arbitrary. We have shown that each of the two problems:

$$(\mathcal{D}_i)_{i=\overline{1,2}} \quad \begin{cases} \mathcal{L}_i \mathbf{v}_i - \nabla p_i - \eta_i \mathbf{v}_i = \mathbf{f}_i + \mathbf{F}_i(\mathbf{u}) + \kappa_i(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i + \lambda_i |\mathbf{u}_i| \mathbf{u}_i & \text{in } \Omega \\ \mathrm{div}\ \mathbf{v}_i = 0 & \text{in } \Omega \\ \mathrm{Tr}\ \mathbf{v}_i = \boldsymbol{\varphi}_i & \text{on } \partial\Omega \end{cases} \tag{23}$$

has a unique solution $(\mathbf{v}_i, p_i) \in H^1_{\mathrm{div}}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$. Let

$$(\boldsymbol{\mathcal{U}}, \mathcal{P}) : H^1_{\mathrm{div}}(\Omega)^{2n} \to H^1_{\mathrm{div}}(\Omega)^{2n} \times \left(L^2(\Omega)/\mathbb{R}\right)^2$$

be operators defined as:

$$\boldsymbol{\mathcal{U}}(\mathbf{u}) = (\boldsymbol{\mathcal{U}}_1(\mathbf{u}), \boldsymbol{\mathcal{U}}_2(\mathbf{u})) \quad \mathcal{P}(\mathbf{u}) = (\mathcal{P}_1(\mathbf{u}), \mathcal{P}_2(\mathbf{u}))$$

where $(\boldsymbol{\mathcal{U}}_i, \mathcal{P}_i)$ are operators mapping $\mathbf{u}$ to solution $(\mathbf{v}_i, p_i)$ of problem $(\mathcal{D}_i)$:

$$(\boldsymbol{\mathcal{U}}_i(\mathbf{u}), \mathcal{P}_i(\mathbf{u})) = (\mathbf{v}_i, p_i) = \mathcal{B}_i(\mathbf{f}_i + \mathbf{F}_i(\mathbf{u}) + \kappa_i(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i + \lambda_i |\mathbf{u}_i| \mathbf{u}_i, \boldsymbol{\varphi}_i), \quad i = \overline{1, 2}.$$

Hence, we can re-write (19) in the following way:

$$\begin{cases} \mathcal{L}_i \boldsymbol{\mathcal{U}}_i(\mathbf{u}) - \nabla \mathcal{P}_i(\mathbf{u}) - \eta_i \boldsymbol{\mathcal{U}}_i(\mathbf{u}) = \mathbf{f}_i + \mathbf{F}_i(\mathbf{u}) + \kappa_i(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i + \lambda_i |\mathbf{u}_i| \mathbf{u}_i & \text{in } \Omega \\ \operatorname{div} \boldsymbol{\mathcal{U}}_i(\mathbf{u}) = 0 & \text{in } \Omega \\ \operatorname{Tr} \boldsymbol{\mathcal{U}}_i(\mathbf{u}) = \boldsymbol{\varphi}_i & \text{on } \partial\Omega. \end{cases}, \quad i = \overline{1,2}.$$

Note that, if there exists $\mathbf{u}^* \in H^1_{\mathrm{div}}(\Omega)^{2n}$ for which $\mathbf{u}^* = \boldsymbol{\mathcal{U}}(\mathbf{u}^*)$, then the pair

$$(\boldsymbol{\mathcal{U}}(\mathbf{u}^*), \mathcal{P}(\mathbf{u}^*)) = (\mathbf{u}^*, \mathcal{P}(\mathbf{u}^*))$$

is a solution for problem (19).

In view of the previously stated estimates, we have:

$$\begin{aligned}\|(\mathcal{U}_i(\mathbf{u}),\mathcal{P}_i(\mathbf{u}))\|_{H^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}} &= \|\mathcal{B}_i(\mathbf{f}_i+\mathbf{F}_i(\mathbf{u})+\kappa_i(\mathbf{u}_i\cdot\nabla)\mathbf{u}_i+\lambda_i|\mathbf{u}_i|\mathbf{u}_i,\varphi_i)\|_{H^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}}\\ &\le C_{*i}\|(\mathbf{f}_i,\varphi_i)\|_{H^{-1}(\Omega)^n\times H^{\frac{1}{2}}(\partial\Omega)^n}+C_iC_{*i}\|\mathbf{u}_i\|^2_{H^1(\Omega)^n}+C_{*i}\|\mathbf{F}_i(\mathbf{u})\|_{H^{-1}(\Omega)^n}.\end{aligned} \tag{24}$$

By denoting

$$C=\max\{C_1,C_2\},\quad C_*=\max\{C_{*1},C_{*2}\}.$$

and taking into account (22), the inequality (24) becomes:

$$\begin{aligned}\|(\mathcal{U}_i(\mathbf{u}),\mathcal{P}_i(\mathbf{u}))\|_{H^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}} \le & C_*\|(\mathbf{f}_i,\varphi_i)\|_{H^{-1}(\Omega)^n\times H_\nu^{\frac{1}{2}}(\partial\Omega)^n}+CC_*\|\mathbf{u}_i\|^2_{H^1(\Omega)^n}\\ &+C_*\left(a_{1i}\|\mathbf{u}_1\|_{H^1(\Omega)^n}+a_{2i}\|\mathbf{u}_2\|_{H^1(\Omega)^n}\right),\quad i=\overline{1,2}.\end{aligned} \tag{25}$$

Let $L \in (0, 1/4)$ be fixed and take:

$$r = \frac{L}{2CC_*}, \quad R = \frac{r - 2r^2 CC_*}{C_*} > 0.$$

Theorem

Assuming that

$$|||(\mathbf{f}, \varphi)|||_{H^{-1}(\Omega)^{2n} \times H_\nu^{\frac{1}{2}}(\partial\Omega)^{2n}} \leq R, \quad a_{ij} \leq {}^{L}/_{4C_*}, \quad 1 \leq i, j \leq 2,$$

the problem (19) *has a unique solution* $(\mathbf{u}, \pi) \in H^1_{\mathrm{div}}(\Omega)^{2n} \times \left(L^2(\Omega)/\mathbb{R}\right)^2$ *with* $|||\mathbf{u}|||_{H^1(\Omega)^{2n}} \leq r$ *and there exists a matrix* $\tilde{C} \in \mathbb{R}^{2\times 2}$ *with positive entries such that:*

$$|||(\mathbf{u}, \pi)|||_{H^1(\Omega)^{2n} \times (L^2(\Omega)/\mathbb{R})^2} \leq \tilde{C} |||(\mathbf{f}, \varphi)|||_{H^{-1}(\Omega)^{2n} \times H^{\frac{1}{2}}(\partial\Omega)^{2n}}.$$

Proof

Let $\mathbf{B}_r$ be the closed ball in $H^1_{\text{div}}(\Omega)^{2n}$ of radius r centered at 0.
First, let us verify that $\mathcal{U}$ is a self operator on $\mathbf{B}_r$. For this, note that, according to (25) and the hypothesis, we have:

$$\|(\mathcal{U}_i(\mathbf{u}), \mathcal{P}_i(\mathbf{u}))\|_{H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}} \le r, \quad \forall \mathbf{u} \in \mathbf{B}_r, i = \overline{1,2}.$$

Second, for any $\mathbf{u}, \mathbf{v} \in \mathbf{B}_r$, the following inequality holds:

$$\begin{aligned}
\|\mathcal{U}_i(\mathbf{u}) - \mathcal{U}_i(\mathbf{v})\|_{H^1(\Omega)^n} &\le C_{*i} \|\kappa_i \left((\mathbf{u}_i \cdot \nabla)\mathbf{u}_i - (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i\right) + \lambda_i \left(|\mathbf{u}_i|\mathbf{u}_i - |\mathbf{v}_i|\mathbf{v}_i\right) + \mathbf{F}_i(\mathbf{u}) - \mathbf{F}_i(\mathbf{v})\|_{H^{-1}(\Omega)^n} \\
&\le 2rCC_* \|\mathbf{u}_i - \mathbf{v}_i\|_{H^1(\Omega)^n} + C_* \left(a_{i1}\|\mathbf{u}_1 - \mathbf{v}_1\|_{H^1(\Omega)^n} + a_{i2}\|\mathbf{u}_2 - \mathbf{v}_2\|_{H^1(\Omega)^n}\right) \\
&\le L\|\mathbf{u}_i - \mathbf{v}_i\|_{H^1(\Omega)^n} + \frac{L}{4}\left(\|\mathbf{u}_1 - \mathbf{v}_1\|_{H^1(\Omega)^n} + \|\mathbf{u}_2 - \mathbf{v}_2\|_{H^1(\Omega)^n}\right), \quad i = \overline{1,2}.
\end{aligned}$$

Equivalently, the previous inequality can be written as:

$$|||\mathcal{U}(\mathbf{u}) - \mathcal{U}(\mathbf{v})|||_{H^1(\Omega)^{2n}} \le \begin{bmatrix} \frac{5L}{4} & \frac{L}{4} \\ \frac{L}{4} & \frac{5L}{4} \end{bmatrix} |||\mathbf{u} - \mathbf{v}|||_{H^1(\Omega)^{2n}}. \qquad (26)$$

Proof (cont.)

Denote by $M \in \mathbb{R}^{2\times 2}$ be the square matrix that appears in (26). Remains to show that M is **convergent to zero**. To this end, observe that:

$$I_2 - M = \frac{1}{4}\begin{bmatrix} 4-5L & -L \\ -L & 4-5L \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

Furthermore, its inverse

$$(I_2 - M)^{-1} = \frac{1}{2(L-1)(3L-2)}\begin{bmatrix} 4-5L & L \\ L & 4-5L \end{bmatrix}$$

has non-negative entries, hence M is **convergent to zero**.

Applying the **Perov fixed point theorem**, we conclude that there exists a unique fixed point $\mathbf{u}^* \in \mathbf{B}_r$ of $\mathcal{U}$, which yields a solution of (19). □

The lid-driven porous cavity flow problem

Problem statement

We study numerically the flow of a **viscous incompressible fluid** in a **square cavity** of length L filled with a **bidisperse porous medium**.

Denote by

$$\mathbf{u}_f(x,y) = (u_f(x,y), v_f(x,y))$$
$$\mathbf{u}_p(x,y) = (u_p(x,y), v_p(x,y))$$

the velocity fields and by $\pi_f(x,y), \pi_p(x,y)$ the pressure fields associated with the $f-$phase and $p-$phase, respectively.

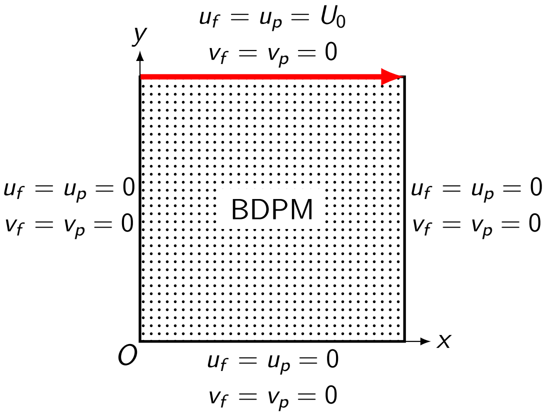


Figure: The geometry of the domain and boundary conditions

We consider a system of two coupled equations of Darcy–Forchheimer–Brinkman type with Dirichlet boundary conditions[21,22]:

$$\begin{cases} \tilde{\mu}_f \Delta \mathbf{u}_f - \nabla \pi_f - \frac{\mu}{K_f}\mathbf{u}_f - \frac{\rho}{\varphi_f^2}(\mathbf{u}_f \cdot \nabla)\mathbf{u}_f - \frac{c_f \rho}{K_f^{1/2}}|\mathbf{u}_f|\mathbf{u}_f - \zeta(\mathbf{u}_f - \mathbf{u}_p) = 0 & \text{in } \Omega \\ \tilde{\mu}_p \Delta \mathbf{u}_p - \nabla \pi_p - \frac{\mu}{K_p}\mathbf{u}_p - \frac{\rho}{\varphi_p^2}(\mathbf{u}_p \cdot \nabla)\mathbf{u}_p - \frac{c_p \rho}{K_p^{1/2}}|\mathbf{u}_p|\mathbf{u}_p - \zeta(\mathbf{u}_p - \mathbf{u}_f) = 0 & \text{in } \Omega \\ \mathrm{div}\ \mathbf{u}_f = 0, \quad \mathrm{div}\ \mathbf{u}_p = 0 & \text{in } \Omega \\ \mathrm{Tr}\ \mathbf{u}_f = (U_0, 0), \quad \mathrm{Tr}\ \mathbf{u}_p = (U_0, 0) & \text{on } \Gamma_{\text{lid}} \\ \mathrm{Tr}\ \mathbf{u}_f = (0,0), \quad \mathrm{Tr}\ \mathbf{u}_p = (0,0) & \text{on } \Gamma \setminus \Gamma_{\text{lid}}, \end{cases} \tag{27}$$

where U_0 is the horizontal velocity of the lid.

[21] D. A. Nield and A. V. Kuznetsov. "A Note on Modeling High Speed Flow in a Bidisperse Porous Medium". 2012.

[22] M. Kohr and R. Precup. "Analysis of Navier-Stokes Models for Flows in Bidisperse Porous Media". 2023.

The lid-driven porous cavity flow problem

Mathematical model: Physical parameters

The physical parameters that appear in (27) are:

- the volume fraction of the $f-$phase: φ_f (macroporosity)
- the porosity of the $p-$phase: φ_p (microporosity)
- the fluid viscosity: μ
- the fluid density: ρ
- the effective viscosities for the two phases: $\tilde{\mu}_f = \mu\varphi_f^{-1}, \tilde{\mu}_p = \mu\varphi_p^{-1}$
- the permeabilities of the two phases: K_f, K_p
- the coefficient for inter-phase momentum transfer: ζ

The Forchheimer coefficients c_f and c_p are dimensionless quantities given by[23]:

$$c_f = \frac{1.75}{\sqrt{150\varphi_f^3}}, \quad c_p = \frac{1.75}{\sqrt{150\varphi_p^3}}.$$

[23] D. A. Nield and A. V. Kuznetsov. "A Note on Modeling High Speed Flow in a Bidisperse Porous Medium". 2012.

We introduce the dimensionless variables:

$$(X,Y)=\frac{(x,y)}{L},\quad \mathbf{U}_{f,p}=(U_{f,p},V_{f,p})=\frac{(u_{f,p},v_{f,p})}{U_0},\quad \Pi_{f,p}=\frac{\pi_{f,p}}{\rho U_0^2},$$

and, by substituting, we obtain the non-dimensional form of (27):

$$\begin{cases} \Delta\mathbf{U}_f-\nabla P_f-\eta_f\mathbf{U}_f-\kappa_f(\mathbf{U}_f\cdot\nabla)\mathbf{U}_f-\lambda_f|\mathbf{U}_f|\mathbf{U}_f-\gamma_f(\mathbf{U}_f-\mathbf{U}_p)=0 & \text{in } \Omega\\ \Delta\mathbf{U}_p-\nabla P_p-\eta_p\mathbf{U}_p-\kappa_p(\mathbf{U}_p\cdot\nabla)\mathbf{U}_p-\lambda_p|\mathbf{U}_p|\mathbf{U}_p-\gamma_p(\mathbf{U}_p-\mathbf{U}_f)=0 & \text{in } \Omega\\ \mathrm{div}\ \mathbf{U}_f=0,\quad \mathrm{div}\ \mathbf{U}_p=0 & \text{in } \Omega\\ \mathrm{Tr}\ \mathbf{U}_f=(1,0),\quad \mathrm{Tr}\ \mathbf{U}_p=(1,0) & \text{on } \Gamma_{\text{lid}}\\ \mathrm{Tr}\ \mathbf{U}_f=(0,0),\quad \mathrm{Tr}\ \mathbf{U}_p=(0,0) & \text{on } \Gamma\setminus\Gamma_{\text{lid}}, \end{cases} \tag{28}$$

where the derivatives are with respect to the new spatial variables (X,Y) and:

$$P_{f,p}=\varphi_{f,p}\mathrm{Re}\Pi_{f,p},\quad \eta_{f,p}=\frac{\varphi_{f,p}}{\mathrm{Da}_{f,p}},\quad \kappa_{f,p}=\frac{\mathrm{Re}}{\varphi_{f,p}},\quad \lambda_{f,p}=\mathrm{Re}\frac{c_{f,p}\varphi_{f,p}}{\mathrm{Da}_{f,p}^{1/2}},\quad \gamma_{f,p}=\frac{\varphi_{f,p}\zeta L^2}{\mu}.$$

We assume $\varphi_f=\varphi_p=\varphi$. Then, $\gamma_f=\gamma_p=\gamma$ is referred to as the **interaction parameter**[24].

[24] B. Straughan. "Bidispersive double diffusive convection". 2018.

The lid-driven porous cavity flow problem

Mathematical model: Streamfunctions

Finally, we introduce the **streamfunctions** Ψ_f and Ψ_p defined as:

$$\frac{\partial \Psi_f}{\partial Y} = U_f, \quad \frac{\partial \Psi_f}{\partial X} = -V_f, \quad \frac{\partial \Psi_p}{\partial Y} = U_f, \quad \frac{\partial \Psi_p}{\partial X} = -V_f,$$

which are computed by solving the following boundary value problem:

$$\begin{cases} \Delta \Psi_f = \frac{\partial U_f}{\partial Y} - \frac{\partial V_f}{\partial X} & \text{in } \Omega \\ \Delta \Psi_p = \frac{\partial U_p}{\partial Y} - \frac{\partial V_p}{\partial X} & \text{in } \Omega \\ \mathrm{Tr}\ \Psi_f = 0, \quad \mathrm{Tr}\ \Psi_p = 0 & \text{on } \Gamma. \end{cases} \tag{29}$$

- The Dirichlet problem (28) is solved numerically using the **finite element** software **FreeFEM**.
- The domain Ω is discretized into a triangular mesh with a N triangles on each of the four segments of the boundary.
- The solution is computed by performing a **fixed point iteration**, which is stopped once the error between consecutive iterates falls bellow $\varepsilon = 10^{-6}$.

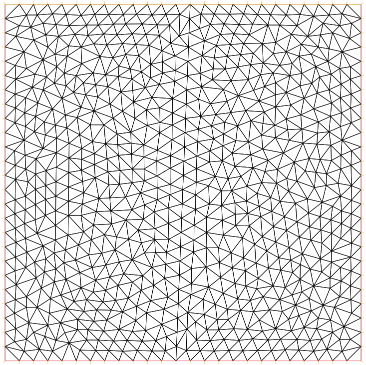


Figure: Mesh for $N = 25$

- The quantities
$$\Psi_f^{\max} = \max_{\Omega} \left|\Psi_f\right|, \quad \Psi_p^{\max} = \max_{\Omega} \left|\Psi_p\right|$$
approach fixed values as N increases.
- The difference between consecutive values for each streamfunction becomes smaller than 10^{-4} from $N = 50$ onward
- Consequently, we choose $N = 50$ in the following simulations.

N	25	50	75	100
$\Psi_f^{\max}$	0.074127	0.074192	0.074205	0.074218
$\Psi_p^{\max}$	0.047714	0.047695	0.047704	0.047700

Table: Maximal values of the streamfunctions at different mesh sizes

($\mathrm{Re} = 100$, $\mathrm{Da}_f = 0.25$, $\mathrm{Da}_p = 0.0025$, $\gamma = 100$)

The lid-driven porous cavity flow problem

Numerical method: Validation

We compare our results with the ones reported by Gutt and Groșan[25] and Gutt[26] for the porous cavity flow problem in the **monodisperse** setting ($\gamma = 0$).

Da	$\varphi = 0.2$		$\varphi = 0.5$	
	Gutt and Groșan 2015	Ours	Gutt 2018	Ours
0.25	0.1139 (0.545, 0.595)	0.1142 (0.552, 0.593)	0.1066 (0.603, 0.673)	0.1062 (0.613, 0.682)
0.025	0.1046 (0.555, 0.600)	0.1049 (0.552, 0.594)	0.0906 (0.642, 0.720)	0.0903 (0.648, 0.721)
0.0025	0.0667 (0.650, 0.670)	0.0668 (0.647, 0.668)	0.0489 (0.692, 0.854)	0.0490 (0.706, 0.860)
0.00025	0.0283 (0.795, 0.905)	0.0284 (0.777, 0.902)	–	

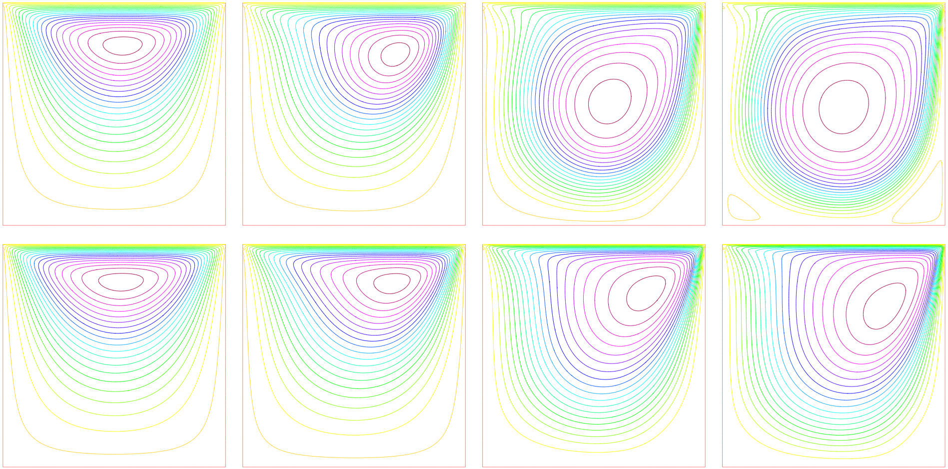
Table: Maximal value of the streamfunction for different Darcy numbers ($\mathrm{Re} = 100$)

[25] R. Gutt and T. Groșan. "On the lid-driven problem in a porous cavity. A theoretical and numerical approach". 2015.

[26] R. Gutt. "BIE and BEM approach for the mixed Dirichlet-Robin boundary value problem for the nonlinear Darcy-Forchheimer-Brinkman system". 2018.

The lid-driven porous cavity flow problem

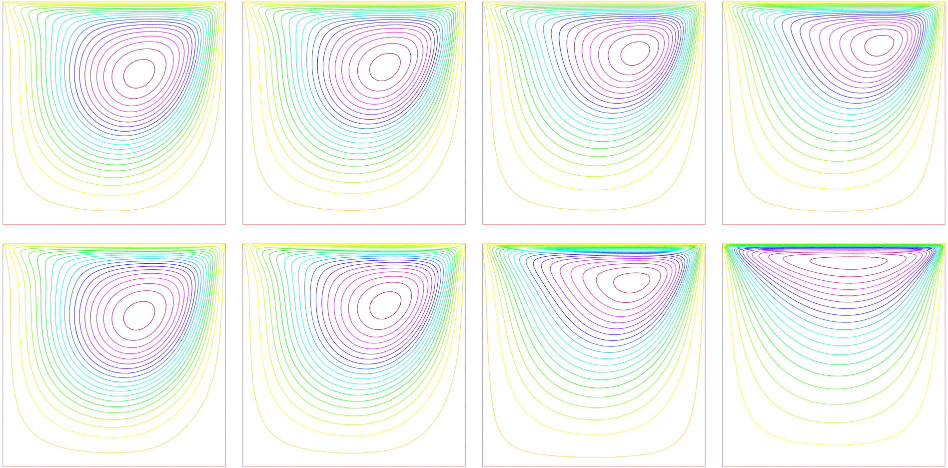
Variation of the Reynolds number



(a) $\mathrm{Re} = 10$ (b) $\mathrm{Re} = 100$ (c) $\mathrm{Re} = 500$ (d) $\mathrm{Re} = 1000$

Figure: Streamlines for $f-$phase (top) and $p-$phase (bottom) at different values of Re

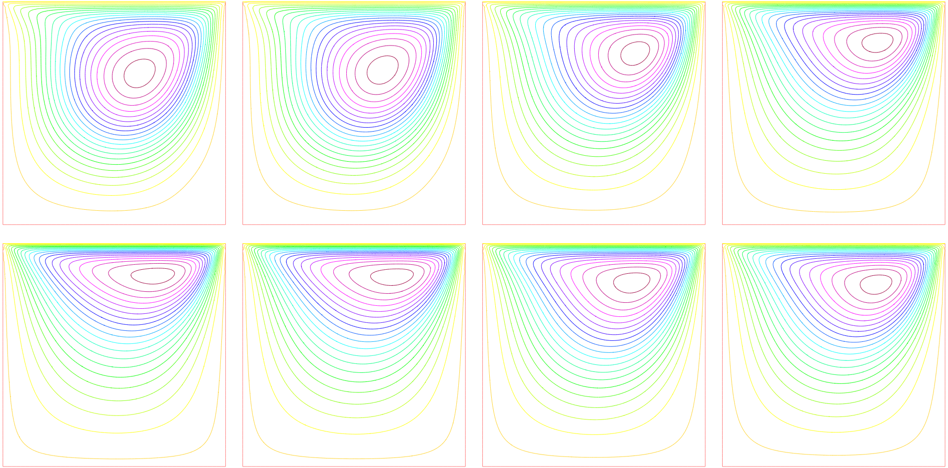
$(\varphi = 0.4,\ \mathrm{Da}_f = 0.25,\ \mathrm{Da}_p = 0.0025,\ \gamma = 100)$



(a) $\mathrm{Da}_p = 0.25$ (b) $\mathrm{Da}_p = 0.025$ (c) $\mathrm{Da}_p = 0.0025$ (d) $\mathrm{Da}_p = 0.00025$

Figure: Streamlines for $f-$phase (top) and $p-$phase (bottom) at different values of Da_p

$(\mathrm{Re} = 100, \varphi = 0.4, \mathrm{Da}_f = 0.25, \gamma = 100)$



(a) $\gamma = 1$ (b) $\gamma = 10$ (c) $\gamma = 100$ (d) $\gamma = 1000$

Figure: Streamlines for $f-$phase (top) and $p-$phase (bottom) at different values of γ

$(\mathrm{Re} = 100, \varphi = 0.4, \mathrm{Da}_f = 0.25, \mathrm{Da}_p = 0.0025)$

Thank you for your attention!

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