Dirichlet problems for the anisotropic Darcy–Forchheimer–Brinkman system and related models

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Outline

- Let $\mathfrak{D} \subset \mathbb{R}^n$ be a bounded Lipschitz domain $(n = 2, 3)$ occupied by a viscous incompressible fluid.
- Let $\mathbf{u} = (u_1, \ldots, u_n)^\top$ be the velocity field and the π be the pressure field.
- Let $\mathbb{E}(\mathbf{u}) = (E_{j\beta}(\mathbf{u}))_{1 \leq j, \beta \leq n}$ be the strain tensor field:

$$\mathbb{E}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right), \quad E_{j\beta}(\mathbf{u}) = \frac{1}{2}\left(\partial_j u_\beta + \partial_\beta u_j\right), \quad 1 \leq j, \beta \leq n.$$

Preliminaries

The stress tensor field $\mathbb{T} = (T_{i\alpha})_{1\leq i,\alpha\leq n}$ of a general (anisotropic) Newtonian fluid satisfies the following constitutive relation:

$$T_{i\alpha}(\mathbf{u},\pi) = -\pi\delta_{i\alpha} + a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}), \quad 1 \leq i,\alpha \leq n \tag{1}$$

where $\mathbb{A} = \left(a_{ij}^{\alpha\beta}\right)_{1\leq i,j,\alpha,\beta\leq n}$ is the viscosity tensor coefficient, whose entries satisfy[1]:

$$a_{ij}^{\alpha\beta} = a_{\alpha j}^{i\beta} = a_{i\beta}^{\alpha j} \quad 1 \leq i,j,\alpha,\beta \leq n. \tag{2}$$

[1] The symmetries of $\mathbb{A}$ are imposed by the symmetries of $\mathbb{T}$ and $\mathbb{E}(\mathbf{u})$

The divergence of the stress tensor field can be written component-wise in the following manner:

$$(\mathrm{div}\ \mathbb{T})_i = \partial_\alpha T_{i\alpha} = \boxed{\partial_\alpha \left(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}) \right)} - \delta_{i\alpha}\partial_\alpha \pi, \quad i = \overline{1, n}. \tag{3}$$

We introduce the following second-order differential operator given in component-wise divergence form as (see, e.g., Kohr et al. 2021[2]):

$$(\mathcal{L}\mathbf{u})_i = \partial_\alpha \left(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}) \right), \quad i = \overline{1, n}. \tag{4}$$

Then:

$$\mathrm{div}\ \mathbb{T} = \mathcal{L}\mathbf{u} - \nabla\pi \tag{5}$$

[2] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "Dirichlet and transmission problems for anisotropic Stokes and Navier-Stokes systems with L^∞ tensor coefficient under relaxed ellipticity condition". 2021.

Remark

In particular, in the isotropic case, the entries of $\mathbb{A}$ are given by:

$$a_{ij}^{\alpha\beta} = \mu(\delta_{\alpha j}\delta_{\beta i} + \delta_{\alpha\beta}\delta_{ij}), \quad 1 \leq i, j, \alpha, \beta \leq n \tag{6}$$

Furthermore, for constant μ, the operator $\mathcal{L}$ has the expression $\mathcal{L}\mathbf{u} = \mu\Delta\mathbf{u}$.

The equations:

$$\begin{cases} \mathcal{L}\mathbf{u} - \nabla\pi - \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} & \text{in } \mathfrak{D} \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \mathfrak{D} \end{cases} \tag{7}$$

determine the (steady-state) anisotropic Navier-Stokes system.

Boundary value problems for the anisotropic Stokes and Navier–Stokes systems were studied by Kohr, Mikhailov, and Wendland[3,4,5] using variational and fixed-point techniques.

[3] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "Dirichlet and transmission problems for anisotropic Stokes and Navier-Stokes systems with L^∞ tensor coefficient under relaxed ellipticity condition". 2021.

[4] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "Non-homogeneous Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

[5] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "On some mixed-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

The Navier–Stokes system (7) can be generalized in order to model **flows in porous media** (see, e.g., Nield and Bejan[6]). In the anisotropic case, such a model is given by the system:

$$\begin{cases} \mathcal{L}\mathbf{u} - \nabla\pi - \eta\mathbf{u} - \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda|\mathbf{u}|\mathbf{u} = \mathbf{f} & \text{in } \mathfrak{D} \\ \operatorname{div}\, \mathbf{u} = 0 & \text{in } \mathfrak{D} \end{cases} \tag{8}$$

where η, κ, λ are parameters which depend on the physical properties of the fluid and the porous medium.

We refer to (8) as the anisotropic Darcy–Forchheimer–Brinkman (DFB) system. By following similar techniques as in Kohr et al. 2022[7,8], we study the non-homogeneous Dirichlet problem for such a system.

[6] Nield, D.A. and Bejan, A. *Convection in Porous Media*. 2017.

[7] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "Non-homogeneous Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

[8] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "On some mixed-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

A **bidisperse porous medium** is a material composed of clusters of large particles that are agglomerations of small particles, or, alternatively, a standard porous medium in which fractures or tunnels have been introduced.

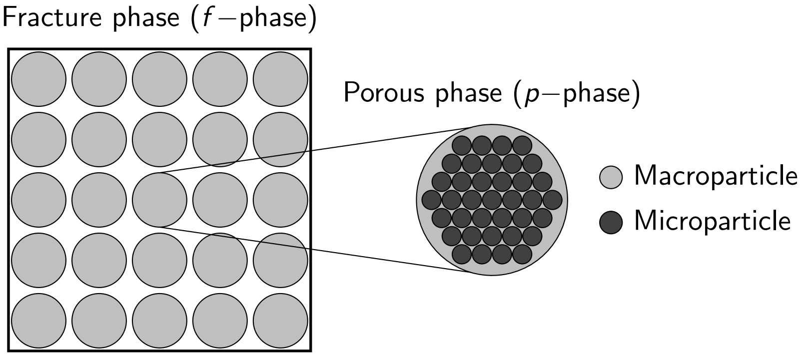


Figure: Square cavity filled with a bidisperse porous medium

Nield and Kuznetsov[9,10] considered the following two-velocity model for steady-state momentum transfer in a **bidisperse porous medium** by extending the Brinkman model from the monodisperse case (* denotes dimensional variables):

$$\begin{cases} \mathbf{G} = \left(\frac{\mu}{K_f}\right)\mathbf{v}_f^* + \zeta(\mathbf{v}_f^* - \mathbf{v}_p^*) - \tilde{\mu}_f(\nabla^*)^2\mathbf{v}_f^* + \frac{c_f\rho}{K_f^{1/2}}|\mathbf{v}_f^*|\mathbf{v}_f^* \\ \mathbf{G} = \left(\frac{\mu}{K_p}\right)\mathbf{v}_p^* + \zeta(\mathbf{v}_p^* - \mathbf{v}_f^*) - \tilde{\mu}_p(\nabla^*)^2\mathbf{v}_p^* + \frac{c_p\rho}{K_p^{1/2}}|\mathbf{v}_p^*|\mathbf{v}_p^* \end{cases}, \tag{9}$$

where:

- $\mathbf{G}$ is the negative of the pressure gradient
- μ, ρ are the fluid viscosity and density
- $\tilde{\mu}_{f,p}$ and $K_{f,p}$ are the effective viscosities and permeabilities of the two phases
- $\mathbf{v}_f^*, \mathbf{v}_p^*$ are velocity fields associated with the two phases
- ζ is the coefficient for momentum transfer between the two phases

[9] Nield, DA and Kuznetsov, AV. "The onset of convection in a bidisperse porous medium". 2006.

[10] Nield, DA and Kuznetsov, AV. "A note on modeling high speed flow in a bidisperse porous medium". 2013.

Kohr and Precup[11] studied the homogeneous Dirichlet problem for the following system of coupled Navier–Stokes-type equations:

$$\begin{cases} -\mu_1 \Delta \mathbf{u}_1 + \eta_1 \mathbf{u}_1 + \kappa_1 (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 + \nabla p_1 = \mathbf{h}_1 - \alpha_1 |\mathbf{u}_1|^{p-1}\mathbf{u}_1 - \gamma_1(\mathbf{u}_1 - \mathbf{u}_2) & \text{in } \mathfrak{D} \\ -\mu_2 \Delta \mathbf{u}_2 + \eta_2 \mathbf{u}_2 + \kappa_2 (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2 + \nabla p_2 = \mathbf{h}_2 - \alpha_2 |\mathbf{u}_2|^{p-1}\mathbf{u}_2 - \gamma_2(\mathbf{u}_2 - \mathbf{u}_1) & \text{in } \mathfrak{D} \\ \operatorname{div} \mathbf{u}_1 = 0, \quad \operatorname{div} \mathbf{u}_2 = 0 & \text{in } \mathfrak{D} \end{cases} \tag{10}$$

with parameters $\mu_i, \gamma_i > 0, \eta_i, \kappa_i, \alpha_i \geq 0, i = 1, 2, p \geq 1$.

By following a similar approach, we study the **non-homogeneous** Dirichlet problem for a system of two coupled **anisotropic DFB** equations.

[11] Kohr, Mirela and Precup, Radu. "Analysis of Navier–Stokes Models for Flows in Bidisperse Porous Media". 2023.

Dirichlet problem for the anisotropic Brinkman system

First, we consider the case when $\kappa = \lambda = 0$ and study the homogeneous Dirichlet problem:

$$\begin{cases} \mathcal{L}\mathbf{u} - \nabla\pi - \eta\mathbf{u} = \mathbf{f} & \text{in } \mathfrak{D} \\ \operatorname{div}\, \mathbf{u} = 0 & \text{in } \mathfrak{D} \\ \gamma_{\mathfrak{D}}\mathbf{u} = 0 & \text{on } \partial\mathfrak{D} \end{cases}, \tag{11}$$

where:

- $\eta \in L^{\infty}(\mathfrak{D})^{n\times n}$ is a matrix-valued parameter endowed with the norm:

$$\|\eta\| = \max_{1\leq i,j\leq n} \|\eta_{ij}\|_{L^{\infty}(\mathfrak{D})} \tag{12}$$

- $\mathbf{f} \in H^{-1}(\mathfrak{D})^n$ is a given distribution
- $\gamma_{\mathfrak{D}} : H^1(\mathfrak{D}) \to H^{\frac{1}{2}}(\partial\mathfrak{D})$ is the trace operator

We assume that:

(A1) $a_{ij}^{\alpha\beta} \in L^\infty(\mathfrak{D})$ for any $1 \leq i, j, \alpha, \beta \leq n$ and we consider:

$$\|\mathbb{A}\| = \max_{1\leq i,j,\alpha,\beta \leq n} \left\| a_{ij}^{\alpha\beta} \right\|_{L^\infty(\mathfrak{D})} \tag{13}$$

(A2) $\mathbb{A}$ satisfies the ellipticity condition only in terms all **symmetric matrices** in $\mathbb{R}^{n\times n}$ with **zero trace**, i.e.,

$$\begin{aligned} \exists C_{\mathbb{A}} > 0 \text{ s.t. } a_{ij}^{\alpha\beta}\xi_{i\alpha}\xi_{j\beta} \geq C_{\mathbb{A}}^{-1}|\boldsymbol{\xi}|^2, \quad \forall \boldsymbol{\xi} = (\xi_{i\alpha})_{1\leq i,\alpha\leq n} \in \mathbb{R}^{n\times n} \\ \text{s.t. } \boldsymbol{\xi} = \boldsymbol{\xi}^\top \text{ and } \sum_{i=1}^{n} \xi_{ii} = 0, \end{aligned} \tag{14}$$

where $|\boldsymbol{\xi}|^2 = \xi_{i\alpha}\xi_{i\alpha}$.

(A3) The parameter η satisfies:

$$\langle \eta \mathbf{w}, \mathbf{w} \rangle_{\mathfrak{D}} \geq 0, \quad \forall \mathbf{w} \in H^1(\mathfrak{D})^n \tag{15}$$

Dirichlet problem for the anisotropic Brinkman system

Mixed variational formulation

We introduce the bilinear forms:

$$a_{\mathbb{A};\mathfrak{D}} : \mathring{H}^1(\mathfrak{D})^n \times \mathring{H}^1(\mathfrak{D})^n \to \mathbb{R}, \quad b_{\mathfrak{D}} : \mathring{H}^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R} \to \mathbb{R}$$

given by:

$$\begin{array}{ll} a_{\mathbb{A};\mathfrak{D}}(\mathbf{u},\mathbf{v}) = \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v}) \right\rangle_{\mathfrak{D}} + \langle \eta \mathbf{u}, \mathbf{v} \rangle_{\mathfrak{D}} \\ b_{\mathfrak{D}}(\mathbf{v}, q) = -\langle \mathrm{div}\ \mathbf{v}, q \rangle_{\mathfrak{D}} \end{array}, \quad \begin{array}{l} \forall \mathbf{u}, \mathbf{v} \in \mathring{H}^1(\mathfrak{D})^n \\ \forall q \in L^2(\mathfrak{D})/\mathbb{R} \end{array} \tag{16}$$

and state the following equivalence result:

Lemma

The Dirichlet problem (11) *is equivalent to the following variational problem:*

$$\begin{cases} a_{\mathbb{A};\mathfrak{D}}(\mathbf{u},\mathbf{v}) + b_{\mathfrak{D}}(\mathbf{v},\pi) = -\langle \mathbf{f}, \mathbf{v} \rangle_{\mathfrak{D}} \\ b_{\mathfrak{D}}(\mathbf{u}, q) = 0 \end{cases}, \quad \begin{array}{l} \forall \mathbf{v} \in \mathring{H}^1(\mathfrak{D})^n \\ \forall q \in L^2(\mathfrak{D})/\mathbb{R} \end{array}. \tag{17}$$

The proof follows similar arguments as the ones presented in Kohr et al. 2021 for the anisotropic Stokes system.

The primary result used in the analysis of problem (17) is the following theorem:

Theorem (Babuška (1973), Brezzi (1974))

Let X and M be two Hilbert spaces and let

$$a : X \times X \to \mathbb{R}, \quad b : X \times M \to \mathbb{R}$$

be two **bounded** *bilinear forms. Let $f \in X'$ and $g \in M'$. Denote by V the subspace of X defined as:*

$$V = \{v \in X \mid b(v, q) = 0, \ \forall q \in M\} \tag{18}$$

...

Theorem (cont.)

Assume that:

1. *The bilinear form a is* **coercive**, *i.e., there exists a constant $c_a > 0$ such that:*

$$a(u, u) \geq c_a \|u\|_X^2, \quad \forall u \in X. \tag{19}$$

2. *The bilinear form b satisfies the* **inf-sup** *(Babuška–Brezzi) condition:*

$$\inf_{q \in M \setminus \{0\}} \sup_{v \in X \setminus \{0\}} \frac{b(v, q)}{\|v\|_X \|q\|_M} \geq c_b \tag{20}$$

for some constant $c_b > 0$.

Theorem (cont.)

Then, for unknowns $(u, p) \in X \times M$*, the mixed variational problem:*

$$\begin{cases} a(u,v) + b(v,p) = f(v), & \forall v \in V \\ b(u,q) = g(q), & \forall q \in M \end{cases} \tag{21}$$

is well-posed, i.e., (21) *has a unique solution* $(u, p) \in X \times M$ *and there exists a constant* $C > 0$ *such that:*

$$\|u\|_X + \|p\|_M \leq C \left(\|f\|_{X'} + \|g\|_{M'} \right). \tag{22}$$

Denote by $\mathring{H}^1_{\mathrm{div}}(\mathfrak{D})^n$ the space:

$$\mathring{H}^1_{\mathrm{div}}(\mathfrak{D})^n = \{\mathbf{w} \in \mathring{H}^1(\mathfrak{D})^n \mid \mathrm{div}\ \mathbf{w} = 0\}$$

Indeed, the bilinear forms $a_{\mathbb{A};\mathfrak{D}}$ and $b_{\mathfrak{D}}$ satisfy the requirements of the previous theorem (see, e.g., Kohr et al. 2021[12]).

Lemma

- *The bilinear form* $a_{\mathbb{A};\mathfrak{D}} : \mathring{H}^1(\mathfrak{D})^n \times \mathring{H}^1(\mathfrak{D})^n \to \mathbb{R}$ *is bounded, i.e., there exists a constant* $c > 0$ *such that:*

$$|a_{\mathbb{A};\mathfrak{D}}(\mathbf{u}, \mathbf{v})| \leq c\|\mathbf{u}\|_{H^1(\mathfrak{D})^n}\|\mathbf{v}\|_{H^1(\mathfrak{D})^n}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathring{H}^1(\mathfrak{D})^n.$$

- *The bilinear form* $a_{\mathbb{A};\mathfrak{D}} : \mathring{H}^1_{\mathrm{div}}(\mathfrak{D})^n \times \mathring{H}^1_{\mathrm{div}}(\mathfrak{D})^n \to \mathbb{R}$ *is coercive, i.e., there exists a constant* $c_a > 0$ *such that:*

$$a_{\mathbb{A};\mathfrak{D}}(\mathbf{u}, \mathbf{u}) \geq c_a\|\mathbf{u}\|^2_{H^1(\mathfrak{D})^n}, \quad \forall \mathbf{u} \in \mathring{H}^1_{\mathrm{div}}(\mathfrak{D})^n.$$

- *The bilinear form* $b_{\mathfrak{D}}$ *is bounded and satisfies the inf-sup condition:*

$$\inf_{q \in L^2(\mathfrak{D})/\mathbb{R}\setminus\{0\}} \sup_{\mathbf{v} \in \mathring{H}^1(\mathfrak{D})^n\setminus\{0\}} \frac{b_{\mathfrak{D}}(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathring{H}^1(\mathfrak{D})^n}\|q\|_{L^2(\mathfrak{D})/\mathbb{R}}} \geq c_b,$$

  for some constant $c_b > 0$.

[12] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "Dirichlet and transmission problems for anisotropic Stokes and Navier-Stokes systems with L^∞ tensor coefficient under relaxed ellipticity condition". 2021.

In view of the equivalence between the homogeneous Dirichlet problem (11) and the mixed variational problem (17) we state the following

Theorem

For any given $\mathbf{f} \in H^{-1}(\mathfrak{D})^n$ *the Dirichlet problem* (11) *has a unique solution* $(\mathbf{u}, \pi) \in \mathring{H}^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}$. *Furthermore, there exists a constant* $C > 0$ *such that:*

$$\|\mathbf{u}\|_{H^1(\mathfrak{D})^n} + \|\pi\|_{L^2(\mathfrak{D})/\mathbb{R}} \leq C\|\mathbf{f}\|_{H^{-1}(\mathfrak{D})^n}.$$

The previous result can be extended to the non-homogeneous Dirichlet problem (following a similar argument as in Kohr et al. 2022[13]):

$$\begin{cases} \mathcal{L}\mathbf{u} - \nabla\pi - \eta\mathbf{u} = \mathbf{f} & \text{in } \mathfrak{D} \\ \operatorname{div}\, \mathbf{u} = 0 & \text{in } \mathfrak{D} \\ \gamma_{\mathfrak{D}}\mathbf{u} = \varphi & \text{on } \partial\mathfrak{D} \end{cases} \tag{23}$$

Theorem

For any given $(\mathbf{f}, \varphi) \in H^{-1}(\mathfrak{D})^n \times H_{\nu}^{\frac{1}{2}}(\partial\mathfrak{D})^n$*, the Dirichlet problem* (23) *has a unique solution* $(\mathbf{u}, \pi) \in H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}$*. Furthermore, there exists a constant* $C > 0$ *such that:*

$$\|\mathbf{u}\|_{H^1(\mathfrak{D})^n} + \|\pi\|_{L^2(\mathfrak{D})/\mathbb{R}} \leq C\left(\|\mathbf{f}\|_{H^{-1}(\mathfrak{D})^n} + \|\varphi\|_{H^{\frac{1}{2}}(\partial\mathfrak{D})^n}\right).$$

[13] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "Non-homogeneous Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

Remark

The solution of the Dirichlet problem (23) *can be represented as* $(\mathbf{u}, \pi) = \mathcal{B}(\mathbf{f}, \varphi)$, *where:*

$$\mathcal{B} : H^{-1}(\mathfrak{D})^n \times H_{\nu}^{\frac{1}{2}}(\partial\mathfrak{D})^n \to H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R},$$

$\mathcal{B} = \mathcal{B}_{\mathfrak{D},\mathbb{A},\eta}$ *is a linear continuous operator.*

Dirichlet problem for the anisotropic DFB system

We now return to the nonlinear case and study the following Dirichlet problem:

$$\begin{cases} \mathcal{L}\mathbf{u} - \nabla\pi - \eta\mathbf{u} - \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda|\mathbf{u}|\mathbf{u} = \mathbf{f} & \text{in } \mathfrak{D} \\ \operatorname{div}\, \mathbf{u} = 0 & \text{in } \mathfrak{D} \\ \gamma_{\mathfrak{D}}\mathbf{u} = \boldsymbol{\varphi} & \text{on } \partial\mathfrak{D} \end{cases}, \tag{24}$$

where

$$\mathbf{f} \in H^{-1}(\mathfrak{D})^n, \quad \boldsymbol{\varphi} \in H_{\nu}^{\frac{1}{2}}(\partial\mathfrak{D})^n \tag{25}$$

are the given data. We assume that:

(A4) $\kappa, \lambda \in L^{\infty}(\mathfrak{D})^{n\times n}$ are matrix-valued parameters with non-negative entries and are endowed with the norms:

$$\|\kappa\| = \max_{1\leq i,j\leq n} \|\kappa_{ij}\|_{L^{\infty}(\mathfrak{D})}, \quad \|\lambda\| = \max_{1\leq i,j\leq n} \|\lambda_{ij}\|_{L^{\infty}(\mathfrak{D})}. \tag{26}$$

To obtain norm estimates for the nonlinear terms in (24) we apply the Hölder inequality along with the following embedding result (consequence of the Sobolev embedding theorem for $n = 2, 3$):

Theorem

For any $r \in [2, 6]$ we have the embedding $H^1(\mathfrak{D})^n \hookrightarrow L^r(\mathfrak{D})$ and there exists a constant c_r such that:

$$\|\mathbf{u}\|_{L^r(\mathfrak{D})^n} \leq c_r \|\mathbf{u}\|_{H^1(\mathfrak{D})^n}, \quad \forall \mathbf{u} \in H^1(\mathfrak{D})^n. \tag{27}$$

We have the following norm estimates (see, e.g., Kohr et al.[14,15]):

Lemma

Let $\mathbf{u}, \mathbf{v} \in H^1(\mathfrak{D})^n$*. Then:*

$$\|(\mathbf{u} \cdot \nabla)\mathbf{v}\|_{H^{-1}(\mathfrak{D})^n} \leq c_4^2 \|\mathbf{u}\|_{H^1(\mathfrak{D})^n} \|\mathbf{v}\|_{H^1(\mathfrak{D})^n}, \tag{28}$$

$$\|(\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{v} \cdot \nabla)\mathbf{v}\|_{H^{-1}(\mathfrak{D})^n} \leq c_4^2 \left(\|\mathbf{u}\|_{H^1(\mathfrak{D})^n} + \|\mathbf{v}\|_{H^1(\mathfrak{D})^n} \right) \|\mathbf{u} - \mathbf{v}\|_{H^1(\mathfrak{D})^n}. \tag{29}$$

Also:

$$\||\mathbf{u}|\mathbf{v}\|_{H^{-1}(\mathfrak{D})^n} \leq c_2 c_4^2 \|\mathbf{u}\|_{H^1(\mathfrak{D})^n} \|\mathbf{v}\|_{H^1(\mathfrak{D})^n}, \tag{30}$$

$$\|(|\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v})\|_{H^{-1}(\mathfrak{D})^n} \leq c_2 c_4^2 \left(\|\mathbf{u}\|_{H^1(\mathfrak{D})^n} + \|\mathbf{v}\|_{H^1(\mathfrak{D})^n} \right) \|\mathbf{u} - \mathbf{v}\|_{H^1(\mathfrak{D})^n}. \tag{31}$$

[14] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "Dirichlet and transmission problems for anisotropic Stokes and Navier-Stokes systems with L^∞ tensor coefficient under relaxed ellipticity condition". 2021.

[15] Kohr, Mirela, Mikhailov, Sergey E, and Wendland, Wolfgang L. "Non-homogeneous Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces". 2022.

Dirichlet problem for the anisotropic DFB system

Let $\mathbf{u} \in H^1_{\mathrm{div}}(\mathfrak{D})^n$. By Theorem 7, the Dirichlet problem:

$$\begin{cases} \mathcal{L}\mathbf{v} - \nabla p - \eta\mathbf{v} = \mathbf{f} + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u} & \text{in } \mathfrak{D} \\ \mathrm{div}\ \mathbf{v} = 0 & \text{in } \mathfrak{D} \\ \gamma_{\mathfrak{D}}\mathbf{v} = \varphi & \text{on } \partial\mathfrak{D} \end{cases} \tag{32}$$

has a unique solution $(\mathbf{v}, p) \in H^1_{\mathrm{div}}(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}$. Let

$$(\mathcal{U}, \mathcal{P}) : H^1_{\mathrm{div}}(\mathfrak{D})^n \to H^1_{\mathrm{div}}(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R} \tag{33}$$

be an operator that maps $\mathbf{u}$ to the unique solution $(\mathbf{v}, p)$ of (32), i.e.,

$$(\mathcal{U}(\mathbf{u}), \mathcal{P}(\mathbf{u})) = (\mathbf{v}, p) = \mathcal{B}\left(\mathbf{f} + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u}, \varphi\right) \tag{34}$$

Dirichlet problem for the anisotropic DFB system

We can re-write (32) as:

$$\begin{cases} \mathcal{L}\boldsymbol{\mathcal{U}}(\mathbf{u}) - \nabla\mathcal{P}(\mathbf{u}) - \eta\boldsymbol{\mathcal{U}}(\mathbf{u}) = \mathbf{f} + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u} & \text{in } \mathfrak{D} \\ \operatorname{div} \boldsymbol{\mathcal{U}}(\mathbf{u}) = 0 & \text{in } \mathfrak{D} \\ \gamma_{\mathfrak{D}}\boldsymbol{\mathcal{U}}(\mathbf{u}) = \boldsymbol{\varphi} & \text{on } \partial\mathfrak{D} \end{cases} . \tag{35}$$

Remark

Observe that if the operator $\boldsymbol{\mathcal{U}}$ *has a fixed point, say* $\mathbf{u}^* = \boldsymbol{\mathcal{U}}(\mathbf{u}^*)$*, then the pair*

$$(\boldsymbol{\mathcal{U}}(\mathbf{u}^*), \mathcal{P}(\mathbf{u}^*)) = (\mathbf{u}^*, \mathcal{P}(\mathbf{u}^*))$$

is a solution of the nonlinear Dirichlet problem (24)*.*

Since the operator $\mathcal{B}$ is bounded, there exists a constant $c_* > 0$ such that:

$$\begin{aligned}\|\boldsymbol{\mathcal{U}}(\mathbf{u}), \mathcal{P}(\mathbf{u})\|_{H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}} &= \|\mathcal{B}\left(\mathbf{f} + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u}, \varphi\right)\|_{H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}} \\ &\leq c_*\left\|\left(\mathbf{f} + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u}, \varphi\right)\right\|_{H^{-1}(\mathfrak{D})^n \times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n} \\ &\leq c_*\left\|(\mathbf{f}, \varphi)\right\|_{H^{-1}(\mathfrak{D})^n \times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n} + c_*\left\|\kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u}\right\|_{H^{-1}(\mathfrak{D})^n}\end{aligned} \tag{36}$$

Moreover:

$$\left\|\kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \lambda|\mathbf{u}|\mathbf{u}\right\|_{H^{-1}(\mathfrak{D})^n} \leq n\left(\|\kappa\|\left\|(\mathbf{u}\cdot\nabla)\mathbf{u}\right\|_{H^{-1}(\mathfrak{D})^n} + \|\lambda\|\left\||\mathbf{u}|\mathbf{u}\right\|_{H^{-1}(\mathfrak{D})^n}\right) \tag{37}$$

With the previously obtained norm estimates, we obtain:

$$\begin{aligned}\|\boldsymbol{\mathcal{U}}(\mathbf{u}), \mathcal{P}(\mathbf{u})\|_{H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}} &\leq c_*\left\|(\mathbf{f}, \varphi)\right\|_{H^{-1}(\mathfrak{D})^n \times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n} + c_* n\left(c_4^2\|\kappa\|\|\mathbf{u}\|^2_{H^1(\mathfrak{D})^n} + c_2 c_4^2\|\lambda\|\|\mathbf{u}\|^2_{H^1(\mathfrak{D})^n}\right) \\ &\leq c_*\left\|(\mathbf{f}, \varphi)\right\|_{H^{-1}(\mathfrak{D})^n \times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n} + c c_*\|\mathbf{u}\|^2_{H^1(\mathfrak{D})^n},\end{aligned} \tag{38}$$

where $c = n\left(c_4^2\|\kappa\| + c_2 c_4^2\|\lambda\|\right)$.

Theorem

Let $\xi = \frac{3}{16cc_*^2}$ *and* $\xi_* = \frac{1}{4cc_*}$. *Assuming that:*

$$\|(\mathbf{f}, \varphi)\|_{H^{-1}(\mathfrak{D})^n \times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n} \leq \xi_*, \tag{39}$$

the Dirichlet problem (24) *has a unique solution*

$$(\mathbf{u}, \pi) \in H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R} \text{ s.t. } \|\mathbf{u}\|_{H^1(\mathfrak{D})^n} \leq \xi.$$

Moreover, there exists a constant $C > 0$ *such that:*

$$\|(\mathbf{u}, \pi)\|_{H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}} \leq C\|(\mathbf{f}, \varphi)\|_{H^{-1}(\mathfrak{D})^n \times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n} \tag{40}$$

Proof

Let $\mathbf{B}_\xi$ be the closed ball in $H^1_{\mathrm{div}}(\mathfrak{D})^n$ of radius η centered at 0:

$$\mathbf{B}_\xi = \{\mathbf{v} \in H^1_{\mathrm{div}}(\mathfrak{D})^n \mid \|\mathbf{v}\|_{H^1(\Omega)^n} \leq \xi\}.$$

We will show that the operator $\boldsymbol{\mathcal{U}} : \mathbf{B}_\xi \to \mathbf{B}_\xi$ has a unique fixed point.
First, let $\mathbf{u} \in \mathbf{B}_\eta$ be arbitrary. Then:

$$\|\boldsymbol{\mathcal{U}}(\mathbf{u}), \mathcal{P}(\mathbf{u})\|_{H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}} \leq \frac{1}{4cc_*} = \xi,$$

hence $\boldsymbol{\mathcal{U}} : \mathbf{B}_\xi \to \mathbf{B}_\xi$ is well-defined, i.e., $\boldsymbol{\mathcal{U}}$ maps the ball $\mathbf{B}_\xi$ to itself.
Next, for any $\mathbf{u}, \mathbf{v} \in \mathbf{B}_\xi$ we have:

$$\|\boldsymbol{\mathcal{U}}(\mathbf{u}) - \boldsymbol{\mathcal{U}}(\mathbf{v})\|_{H^1(\mathfrak{D})^n} \leq c_* \|\kappa\left((\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{v} \cdot \nabla)\mathbf{v}\right) + \lambda\left(|\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v}\right)\|_{H^{-1}(\mathfrak{D})^n} \tag{41}$$

Proof

According to the previously obtained norm estimates:

$$\begin{aligned}\|\mathcal{U}(\mathbf{u})-\mathcal{U}(\mathbf{v})\|_{H^1(\mathfrak{D})^n} &\leq c_* n\Big[\|\kappa\|\|(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{H^{-1}(\mathfrak{D})^n}+\|\lambda\|\||\mathbf{u}|\mathbf{u}-|\mathbf{v}|\mathbf{v}\|_{H^{-1}(\mathfrak{D})^n}\Big]\\ &\leq c_* n\left(c_4^2\|\kappa\|+c_2c_4^2\|\alpha\|\right)\left(\|\mathbf{u}\|_{H^1(\mathfrak{D})^n}+\|\mathbf{v}\|_{H^1(\Omega)^n}\right)\|\mathbf{u}-\mathbf{v}\|_{H^1(\mathfrak{D})^n}\end{aligned} \tag{41}$$

which leads to:

$$\|\mathcal{U}(\mathbf{u})-\mathcal{U}(\mathbf{v})\|_{H^1(\mathfrak{D})^n} \leq 2\xi cc_*\|\mathbf{u}-\mathbf{v}\|_{H^1(\mathfrak{D})^n}=\frac{1}{2}\|\mathbf{u}-\mathbf{v}\|_{H^1(\mathfrak{D})^n}, \tag{42}$$

which means that the operator $\mathcal{U}:\mathbf{B}_\xi\to\mathbf{B}_\xi$ is a contraction.
In view of the Banach contraction principle, there exists a unique fixed point $\mathbf{u}^*\in\mathbf{B}_\xi$ of $\mathcal{U}$.

Proof

Let $\pi^* = \mathcal{P}(\mathbf{u}^*)$. Since $\mathbf{u}^* \in \mathbf{B}_\xi$, i.e., $\|\mathbf{u}^*\|_{H^1(\mathfrak{D})^n} \le \xi = \frac{1}{4cc_*}$, we have:

$$\|(\mathbf{u}^*, \pi^*)\|_{H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}} \le c_* \|(\mathbf{f}, \varphi)\|_{H^{-1}(\mathfrak{D})^n \times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n} + \frac{1}{4}\|\mathbf{u}^*\|_{H^1(\mathfrak{D})^n}, \tag{41}$$

hence:

$$\|\mathbf{u}^*\|_{H^1(\mathfrak{D})^n} + \frac{4}{3}\|\pi^*\|_{L^2(\mathfrak{D})/\mathbb{R}} \le \frac{4c_*}{3}\|(\mathbf{f}, \varphi)\|_{H^{-1}(\mathfrak{D})^n \times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n}. \tag{42}$$

Finally:

$$\|(\mathbf{u}^*, \pi^*)\|_{H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}} \le \|\mathbf{u}^*\|_{H^1(\mathfrak{D})^n} + \frac{4}{3}\|\pi^*\|_{L^2(\mathfrak{D})/\mathbb{R}} \le \frac{4c_*}{3}\|(\mathbf{f}, \varphi)\|_{H^{-1}(\mathfrak{D})^n \times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n}. \tag{43}$$

□

Dirichlet problem for coupled anisotropic DFB equations

We consider the Dirichlet problem for a system of two **coupled anisotropic DFB equations**:

$$\begin{cases} \mathcal{L}_1\mathbf{u}_1 - \nabla\pi_1 - \eta_1\mathbf{u}_1 - \kappa_1(\mathbf{u}_1\cdot\nabla)\mathbf{u}_1 - \lambda_1|\mathbf{u}_1|\mathbf{u}_1 = \mathbf{f}_1 + \mathbf{F}_1(\mathbf{u}_1,\mathbf{u}_2) & \text{in } \mathfrak{D} \\ \mathcal{L}_2\mathbf{u}_2 - \nabla\pi_2 - \eta_2\mathbf{u}_2 - \kappa_2(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2 - \lambda_2|\mathbf{u}_2|\mathbf{u}_2 = \mathbf{f}_2 + \mathbf{F}_2(\mathbf{u}_1,\mathbf{u}_2) & \text{in } \mathfrak{D} \\ \operatorname{div}\, \mathbf{u}_1 = 0, \quad \operatorname{div}\, \mathbf{u}_2 = 0 & \text{in } \mathfrak{D} \\ \gamma_{\mathfrak{D}}\mathbf{u}_1 = \boldsymbol{\varphi}_1, \quad \gamma_{\mathfrak{D}}\mathbf{u}_2 = \boldsymbol{\varphi}_2 & \text{on } \partial\mathfrak{D} \end{cases} \tag{44}$$

where:

$$\mathbf{f} = (\mathbf{f}_1,\mathbf{f}_2) \in H^{-1}(\mathfrak{D})^{2n}, \quad \boldsymbol{\varphi} = (\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2) \in H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\mathfrak{D})^{2n} \tag{45}$$

are the given data, and:

$$\mathbf{F} = (\mathbf{F}_1,\mathbf{F}_2), \quad \mathbf{F}_i : H^1(\mathfrak{D})^{2n} \to H^{-1}(\mathfrak{D})^n, \quad i = \overline{1,2}. \tag{46}$$

We assume that:

(A5) The mapping $\mathbf{F}$ satisfies:

$$\mathbf{F}_i(0) = 0, \quad i = \overline{1,2}. \tag{47}$$

In particular, for modeling flow in **bidisperse porous media**, we are interested in $\mathbf{F}$ of the form:

$$\begin{matrix} \mathbf{F}_1(\mathbf{u}_1, \mathbf{u}_2) = \gamma_1(\mathbf{u}_1 - \mathbf{u}_2) \\ \mathbf{F}_2(\mathbf{u}_1, \mathbf{u}_2) = \gamma_2(\mathbf{u}_2 - \mathbf{u}_1) \end{matrix}, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in H^1_{\mathrm{div}}(\mathfrak{D})^n. \tag{48}$$

for some parameters $\gamma_1, \gamma_2 \in L^\infty(\mathfrak{D})^{n\times n}$ with non-negative entries.

Let (X, d) be a metric space. We introduce the following vector-valued metric on the product space $X^k = \underbrace{X \times X \times \cdots \times X}_{k \text{ times}}$ (see Precup[16], also Petrușel et al.[17]):

$$D : X^k \to [0, \infty)^k, \quad D(x, y) = \begin{pmatrix} d(x_1, y_1) \\ d(x_2, y_2) \\ \vdots \\ d(x_k, y_k) \end{pmatrix}, \quad \forall \begin{matrix} x = (x_1, \ldots, x_k) \\ y = (y_1, \ldots, y_k) \end{matrix} \in X^k. \tag{49}$$

We refer to the space (X^k, D) as a generalized metric space.

[16] Precup, Radu. "The role of matrices that are convergent to zero in the study of semilinear operator systems". 2009.

[17] Petrușel, Adrian, Petrușel, Gabriela, and Urs, Cristina. "Vector-valued metrics, fixed points and coupled fixed points for nonlinear operators". 2013.

Let (X, d_X), (Y, d_Y) be metric spaces and $F : X^k \to Y^k$.

Definition

We say that F is a **generalized Lipschitz mapping** if there exists a matrix $M \in \mathbb{R}^{k\times k}$ with non-negative entries such that:

$$D_{Y^k}(F(x), F(y)) \le M D_{X^k}(x, y), \quad \forall \begin{matrix} x = (x_1, \dots, x_k) \\ y = (y_1, \dots, y_k) \end{matrix} \in X^k. \tag{50}$$

Definition

If $X \equiv Y$, we say that F ia a **generalized contraction** if the matrix M is convergent to zero, i.e.,

$$\lim_{p\to\infty} M^p = 0.$$

Regarding matrices that are convergent to zero we state the following characterization result (see, e.g., Precup 2009):

Lemma

Let $M \in \mathbb{R}^{k\times k}$ be a matrix with non-negative entries. The following statements are equivalent:

- *M is convergent to zero*
- *$I - M$ is non-singular and $(I - M)^{-1}$ has non-negative elements*
- *$|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I_k) = 0$*

The following theorem generalizes the Banach contraction principle to the setting of generalized metric spaces[18]:

Theorem (Perov, 1966)

Let (X, d) be a complete metric space and $F : X^k \to X^k$ be a generalized contraction. Then, there exists a unique fixed point $x^* \in X^k$ of F.

In particular, if $(X, \|\cdot\|_X)$ is a normed space, we consider the following vector-valued norm on X^k:

$$|||\cdot||| : X^k \to [0, \infty)^k, \quad |||x|||_{X^k} = \begin{pmatrix} \|x_1\|_X \\ \|x_2\|_X \\ \vdots \\ \|x_k\|_X \end{pmatrix}, \quad \forall x = (x_1, \dots, x_k) \in X^k. \tag{51}$$

[18] Perov, Anatoliy Ivanovich and Kibenko, AV. "On a certain general method for investigation of boundary value problems". 1966.

Returning to the study of the Dirichlet problem (44), we assume that:

(A6) The mapping $\mathbf{F}$ satisfies the following Lipschitz condition:

$$|||\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v})|||_{H^{-1}(\mathfrak{D})^{2n}} \leq A|||\mathbf{u} - \mathbf{v}|||_{H^1_{\mathrm{div}}(\mathfrak{D})^{2n}} \tag{52}$$

for some matrix $A = (a_{ij})_{1\leq i,j\leq 2} \in \mathbb{R}^{2\times 2}$ with non-negative entries.

Immediately, for any $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in H^1(\mathfrak{D})^{2n}$ we deduce the following bound for $\mathbf{F}$:

$$|||\mathbf{F}(\mathbf{u})|||_{H^{-1}(\mathfrak{D})^{2n}} = |||\mathbf{F}(\mathbf{u}) - \mathbf{F}(0)|||_{H^{-1}(\mathfrak{D})^{2n}} \leq A|||\mathbf{u}|||_{H^1(\mathfrak{D})^{2n}}. \tag{53}$$

Dirichlet problem for coupled anisotropic DFB equations

Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in H^1_{\rm div}(\mathfrak{D})^{2n}$ be arbitrary. By Theorem 7, each of the following Dirichlet problems ($i = \overline{1,2}$):

$$(D_i) \quad \begin{cases} \mathcal{L}_i \mathbf{v}_i - \nabla p_i - \eta_i \mathbf{v}_i = \mathbf{f}_i + \mathbf{F}_i(\mathbf{u}) + \kappa_i(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i + \lambda_i |\mathbf{u}_i| \mathbf{u}_i & \text{in } \mathfrak{D} \\ {\rm div}\ \mathbf{v}_i = 0 & \text{in } \mathfrak{D} \\ \gamma_{\mathfrak{D}} \mathbf{v}_i = \boldsymbol{\varphi}_i & \text{on } \partial\mathfrak{D} \end{cases} \tag{54}$$

has a unique solution $(\mathbf{v}_i, p_i) \in H^1_{\rm div}(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}$. Let

$$(\boldsymbol{\mathcal{U}}_i, \mathcal{P}_i) : H^1_{\rm div}(\Omega)^{2n} \to H^1_{\rm div}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$$

be operators that map $\mathbf{u}$ to the unique solution $(\mathbf{v}_i, p_i)$ of the Dirichlet problem D_i:

$$(\boldsymbol{\mathcal{U}}_i(\mathbf{u}), \mathcal{P}_i(\mathbf{u})) = (\mathbf{v}_i, p_i) = \mathcal{B}_i(\mathbf{f}_i + \mathbf{F}_i(\mathbf{u}) + \kappa_i(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i + \lambda_i |\mathbf{u}_i| \mathbf{u}_i, \boldsymbol{\varphi}_i). \tag{55}$$

Dirichlet problem for coupled anisotropic DFB equations

Hence, we can write D_i as:

$$\begin{cases} \mathcal{L}_i \boldsymbol{\mathcal{U}}_i(\mathbf{u}) - \nabla \mathcal{P}_i(\mathbf{u}) - \eta_i \boldsymbol{\mathcal{U}}_i(\mathbf{u}) = \mathbf{f}_i + \mathbf{F}_i(\mathbf{u}) + \kappa_i(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i + \lambda_i |\mathbf{u}_i| \mathbf{u}_i & \text{in } \mathfrak{D} \\ \operatorname{div} \boldsymbol{\mathcal{U}}_i(\mathbf{u}) = 0 & \text{in } \mathfrak{D} \\ \gamma_{\mathfrak{D}} \boldsymbol{\mathcal{U}}_i(\mathbf{u}) = \boldsymbol{\varphi}_i & \text{on } \partial\mathfrak{D} \end{cases} \quad (56)$$

Let

$$(\boldsymbol{\mathcal{U}}, \mathcal{P}) : H^1_{\operatorname{div}}(\mathfrak{D})^{2n} \to H^1_{\operatorname{div}}(\mathfrak{D})^{2n} \times \left(L^2(\mathfrak{D})/\mathbb{R}\right)^2 \quad (57)$$

be operators defined as:

$$\begin{matrix} \boldsymbol{\mathcal{U}}(\mathbf{u}) = (\boldsymbol{\mathcal{U}}_1(\mathbf{u}), \boldsymbol{\mathcal{U}}_2(\mathbf{u})) = (\mathbf{v}_1, \mathbf{v}_2) \\ \mathcal{P}(\mathbf{u}) = (\mathcal{P}_1(\mathbf{u}), \mathcal{P}_2(\mathbf{u})) = (p_1, p_2) \end{matrix}, \quad \forall \mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in H^1_{\operatorname{div}}(\mathfrak{D})^{2n}. \quad (58)$$

Remark

Observe that if the operator $\boldsymbol{\mathcal{U}}$ *has a fixed point, say* $\mathbf{u}^* = (\mathbf{u}_1^*, \mathbf{u}_2^*) \in H^1_{\operatorname{div}}(\mathfrak{D})^{2n}$ *such that* $\mathbf{u}^* = \boldsymbol{\mathcal{U}}(\mathbf{u}^*)$, *then*

$$(\boldsymbol{\mathcal{U}}(\mathbf{u}^*), \mathcal{P}(\mathbf{u}^*)) = (\mathbf{u}^*, \mathcal{P}(\mathbf{u}^*)) = ((\mathbf{u}_1^*, \mathbf{u}_2^*), (\pi_1^*, \pi_2^*))$$

is a solution of the Dirichlet problem (44).

According to previous results:

$$\begin{aligned}\|\mathcal{U}_i(\mathbf{u}),\mathcal{P}_i(\mathbf{u})\|_{H^1(\mathfrak{D})^n\times L^2(\mathfrak{D})/\mathbb{R}} &= \|\mathcal{B}_i(\mathbf{f}_i+\mathbf{F}_i(\mathbf{u})+\kappa_i(\mathbf{u}_i\cdot\nabla)\mathbf{u}_i+\lambda_i|\mathbf{u}_i|\mathbf{u}_i,\varphi_i)\|_{H^1(\mathfrak{D})^n\times L^2(\mathfrak{D})/\mathbb{R}}\\ &\le c_{*i}\left(\|(\mathbf{f}_i,\varphi_i)\|_{H^{-1}(\mathfrak{D})^n\times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n}+\|\mathbf{F}_i(\mathbf{u})\|_{H^{-1}(\mathfrak{D})^n}+\|\kappa_i(\mathbf{u}_i\cdot\nabla)\mathbf{u}_i+\lambda_i|\mathbf{u}_i|\mathbf{u}_i\|_{H^{-1}(\mathfrak{D})^n}\right)\\ &\le c_{*i}\left(\|(\mathbf{f}_i,\varphi_i)\|_{H^{-1}(\mathfrak{D})^n\times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n}+\|\mathbf{F}_i(\mathbf{u})\|_{H^{-1}(\mathfrak{D})^n}\right)+c_ic_{*i}\|\mathbf{u}_i\|^2_{H^1(\mathfrak{D})^n}\end{aligned}$$

Denote:

$$c=\max_{i=1,2}\{c_i\},\quad c_*=\max_{i=1,2}\{c_{*i}\}.$$

Then, in view of the properties of **F**:

$$\begin{aligned}\|\mathcal{U}_i(\mathbf{u}),\mathcal{P}_i(\mathbf{u})\|_{H^1(\mathfrak{D})^n\times L^2(\mathfrak{D})/\mathbb{R}} \le & c_*\|(\mathbf{f}_i,\varphi_i)\|_{H^{-1}(\mathfrak{D})^n\times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^n}+cc_*\|\mathbf{u}_i\|^2_{H^1(\mathfrak{D})^n}+\\ & c_*\left(a_{1i}\|\mathbf{u}_1\|_{H^1(\mathfrak{D})^n}+a_{2i}\|\mathbf{u}_2\|_{H^1(\mathfrak{D})^n}\right)\end{aligned} \tag{59}$$

Theorem

Let $\xi = \frac{1}{4cc_*}, \xi_* = \frac{1}{8cc_*^2}$. *Assume that:*

$$\max_{1\le i,j\le 2} a_{ij} \le \frac{1}{8c_*}, \quad \left|\left|\left|(\mathbf{f},\varphi)\right|\right|\right|_{H^{-1}(\mathfrak{D})^{2n}\times H_{\nu}^{\frac{1}{2}}(\partial\mathfrak{D})^{2n}} \le \xi_*. \tag{60}$$

Then, the Dirichlet problem (44) *has a unique solution*

$$(\mathbf{u},\pi) \in H^1(\mathfrak{D})^{2n} \times \left(L^2(\mathfrak{D})/\mathbb{R}\right)^2 \ \textit{s.t.}\ \left|\left|\left|\mathbf{u}\right|\right|\right|_{H^1(\mathfrak{D})^{2n}} \le \xi.$$

Moreover, there exists a constant $C > 0$ *such that:*

$$\left|\left|\left|(\mathbf{u},\pi)\right|\right|\right|_{H^1(\mathfrak{D})^{2n}\times(L^2(\mathfrak{D})/\mathbb{R})^2} \le C\left|\left|\left|(\mathbf{f},\varphi)\right|\right|\right|_{H^{-1}(\mathfrak{D})^{2n}\times H_{\nu}^{\frac{1}{2}}(\partial\mathfrak{D})^{2n}}. \tag{61}$$

Proof

Let $\mathbf{B}_\xi$ be the closed ball in $H^1(\mathfrak{D})^{2n}$ of radius ξ centered at 0:

$$\mathbf{B}_\xi = \left\{ \mathbf{v} \in H^1(\mathfrak{D})^{2n} \mid |||\mathbf{v}|||_{H^1(\mathfrak{D})^{2n}} \leq \xi \right\}. \tag{62}$$

We will show that the operator $\boldsymbol{\mathcal{U}} : \mathbf{B}_\xi \to \mathbf{B}_\xi$ has a unique fixed point $\mathbf{u}^* \in \mathbf{B}_\xi$.
Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{B}_\xi$ be arbitrary. Then, for $i = \overline{1,2}$, we have:

$$\|\boldsymbol{\mathcal{U}}_i(\mathbf{u}), \mathcal{P}_i(\mathbf{u})\|_{H^1(\mathfrak{D})^n \times L^2(\mathfrak{D})/\mathbb{R}} \leq c_*\xi_* + cc_*\xi^2 + c_*\xi(a_{1i} + a_{2i}) = \frac{1}{4cc_*} = \xi, \tag{63}$$

therefore the operator $\boldsymbol{\mathcal{U}} : \mathbf{B}_\xi \to \mathbf{B}_\xi$ is well-defined.

Proof

Moreover, for any $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ and $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in H^1(\mathfrak{D})^{2n}$ we have $(i = \overline{1,2})$:

$$\|\mathcal{U}_i(\mathbf{u}) - \mathcal{U}_i(\mathbf{v})\|_{H^1(\mathfrak{D})^n} \leq c_{*i}\|\kappa_i\left((\mathbf{u}_i \cdot \nabla)\mathbf{u}_i - (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i\right) + \lambda_i\left(|\mathbf{u}_i|\mathbf{u}_i \; - |\mathbf{v}_i|\mathbf{v}_i\right)\|_{H^{-1}(\mathfrak{D})^n} + c_{*i}\|\mathbf{F}_i(\mathbf{u}) - \mathbf{F}_i(\mathbf{v})\|_{H^{-1}(\mathfrak{D})^n},$$

which, taking into account the assumptions on $\mathbf{F}$, becomes:

$$\begin{aligned} \|\mathcal{U}_i(\mathbf{u}) - \mathcal{U}_i(\mathbf{v})\|_{H^1(\mathfrak{D})^n} &\leq 2\xi c c_*\|\mathbf{u}_i - \mathbf{v}_i\| + c_*(a_{1i}\|\mathbf{u}_1 - \mathbf{v}_1\| + a_{2i}\|\mathbf{u}_2 - \mathbf{v}_2\|) \\ &\leq \frac{1}{2}\|\mathbf{u}_i - \mathbf{v}_i\| + \frac{1}{8}\left(\|\mathbf{u}_1 - \mathbf{v}_1\| + \|\mathbf{u}_2 - \mathbf{v}_2\|\right) \end{aligned} \tag{62}$$

hence:

$$\begin{pmatrix} \|\mathcal{U}_1(\mathbf{u}) - \mathcal{U}_1(\mathbf{v})\|_{H^1(\mathfrak{D})^n} \\ \|\mathcal{U}_2(\mathbf{u}) - \mathcal{U}_2(\mathbf{v})\|_{H^1(\mathfrak{D})^n} \end{pmatrix} \leq \begin{pmatrix} \frac{5}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{8} \end{pmatrix} \begin{pmatrix} \|\mathbf{u}_1 - \mathbf{v}_1\|_{H^1(\mathfrak{D})^n} \\ \|\mathbf{u}_2 - \mathbf{v}_2\|_{H^1(\mathfrak{D})^n} \end{pmatrix} \tag{63}$$

Proof

Let $M = \begin{pmatrix} \frac{5}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{8} \end{pmatrix} \in \mathbb{R}^{2\times 2}$. We can write the previous inequality as:

$$|||\boldsymbol{\mathcal{U}}(\mathbf{u}) - \boldsymbol{\mathcal{U}}(\mathbf{v})|||_{H^1(\mathfrak{D})^{2n}} \leq M|||\mathbf{u} - \mathbf{v}|||_{H^1(\mathfrak{D})^{2n}}. \tag{62}$$

Since the matrix $I_2 - M$ is non-singular and its inverse

$$(I_2 - M)^{-1} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

has non-negative entries, it follows that the matrix M is convergent to zero. In view of the Perov fixed point theorem, there exists a unique fixed point $\mathbf{u}^* \in \mathbf{B}_\xi$ of $\boldsymbol{\mathcal{U}}$.

Proof

Let $\pi^* = \mathcal{P}(\mathbf{u}^*)$. Since $\mathbf{u}^* \in \mathbf{B}_\xi$, i.e., $|||\mathbf{u}^*|||_{H^1(\mathfrak{D})^{2n}} \le \xi = \frac{1}{4cc_*}$, we have:

$$|||(\mathbf{u}^*, \pi^*)|||_{H^1(\mathfrak{D})^{2n}\times(L^2(\mathfrak{D})/\mathbb{R})^2} \le c_*|||(\mathbf{f}, \varphi)|||_{H^{-1}(\mathfrak{D})^{2n}\times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^{2n}} + \frac{1}{2}|||\mathbf{u}^*|||_{H^1(\mathfrak{D})^{2n}}$$

hence:

$$|||\mathbf{u}^*|||_{H^1(\mathfrak{D})^{2n}} + 2|||\pi^*|||_{(L^2(\mathfrak{D})/\mathbb{R})^2} \le 2c^*|||(\mathbf{f}, \varphi)|||_{H^{-1}(\mathfrak{D})^{2n}\times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^{2n}}$$

Finally:

$$|||(\mathbf{u}^*, \pi^*)|||_{H^1(\mathfrak{D})^{2n}\times(L^2(\mathfrak{D})/\mathbb{R})^2} \le |||\mathbf{u}^*|||_{H^1(\mathfrak{D})^{2n}} + 2|||\pi^*|||_{(L^2(\mathfrak{D})/\mathbb{R})^2} \le 2c^*|||(\mathbf{f}, \varphi)|||_{H^{-1}(\mathfrak{D})^{2n}\times H_\nu^{\frac{1}{2}}(\partial\mathfrak{D})^{2n}}$$

□

The lid-driven cavity flow problem

We study numerically the flow of viscous incompressible a fluid in a square cavity of length L filled with a monodisperse/bidisperse porous medium.
Let $\mathbf{u}(x, y) = (u(x, y), v(x, y))$ be the velocity field and $\pi(x, y)$ be the pressure field.

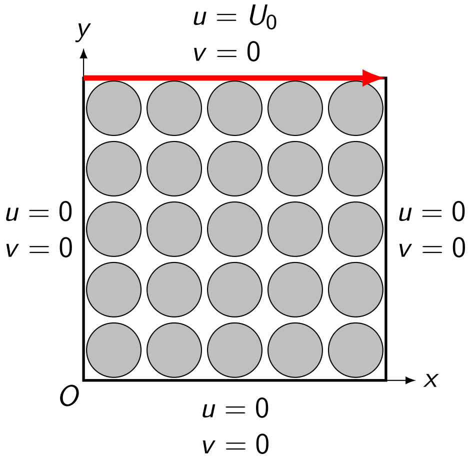


Figure: The geometry of the flow domain and boundary conditions

The model governing the steady state of the flow is given by the following DFB system ([19,20]):

$$\begin{cases} \mu\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)-\varphi\dfrac{\partial \pi}{\partial x}-\dfrac{\mu\varphi}{K}u-\dfrac{\rho}{\varphi}\left(u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}\right)-\varphi c u\sqrt{u^2+v^2}=0 \\ \mu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)-\varphi\dfrac{\partial \pi}{\partial y}-\dfrac{\mu\varphi}{K}v-\dfrac{\rho}{\varphi}\left(u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}\right)-\varphi c v\sqrt{u^2+v^2}=0 \\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \end{cases} \tag{62}$$

where:

- ρ and μ is the density and viscosity of the fluid
- φ and K are the porosity and permeability of the porous medium
- c is an empirical coefficient: $c = \frac{1.75\rho}{\sqrt{150\varphi K}}$.

[19] Nield, D.A. and Bejan, A. *Convection in Porous Media*. 2017.

[20] Gutt, Robert and Groșan, Teodor. "On the lid-driven problem in a porous cavity. A theoretical and numerical approach". 2015.

We introduce the dimensionless variables:

$$X = \frac{x}{L}, \quad Y = \frac{y}{L}, \quad U = \frac{u}{U_0}, \quad V = \frac{v}{U_0}, \quad \Pi = \frac{\pi}{\rho U_0^2} \tag{63}$$

and we denote:

$$\eta = \frac{\varphi}{\mathrm{Da}}, \quad \kappa = \frac{\mathrm{Re}}{\varphi}, \quad \lambda = \varphi C \mathrm{Re}, \quad C = \frac{1.75}{\sqrt{150\varphi \mathrm{Da}}}, \quad P = \varphi \mathrm{Re} \Pi, \tag{64}$$

where

$$\mathrm{Re} = \frac{U_0 L \rho}{\mu}, \quad \mathrm{Da} = \frac{K}{L^2} \tag{65}$$

are the Reynolds number and Darcy number.

We obtain the following non-dimensional form of the DFB system (62):

$$\begin{cases} \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} - \frac{\partial P}{\partial X} - \eta U - \kappa \left(U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} \right) - \lambda U \sqrt{U^2 + V^2} = 0 \\ \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} - \frac{\partial P}{\partial Y} - \eta V - \kappa \left(U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} \right) - \lambda V \sqrt{U^2 + V^2} = 0 \\ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \end{cases} \tag{66}$$

Remark

In particular, by lifting the porous medium assumption (letting $\varphi = 1$ and $\mathrm{Da} \to \infty$, which yields $\eta = \lambda = 0$, $\kappa = \mathrm{Re}$), we obtain the Navier–Stokes system.

We introduce the streamfunction Ψ defined by:

$$\frac{\partial \Psi}{\partial Y} = U, \quad \frac{\partial \Psi}{\partial X} = -V \tag{67}$$

and the vorticity field:

$$\Omega = \frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y}. \tag{68}$$

By substituting (67) in (68) we obtain the following relationship between Ψ and Ω:

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = -\Omega \tag{69}$$

By differentiating the first relation in (66) with respect to Y, the second one with respect to X, subtracting, and coupling with (69), we obtain the streamfunction–vorticity formulation of the system (66) (cf. Gutt and Groșan 2015):

$$\begin{cases} \dfrac{\partial^2\Omega}{\partial X^2} + \dfrac{\partial^2\Omega}{\partial Y^2} - \eta\Omega - \kappa\left(\dfrac{\partial\Psi}{\partial Y}\dfrac{\partial\Omega}{\partial X} - \dfrac{\partial\Psi}{\partial X}\dfrac{\partial\Omega}{\partial Y}\right) + \\ \qquad \dfrac{\lambda}{\sqrt{\left(\frac{\partial\Psi}{\partial X}\right)^2 + \left(\frac{\partial\Psi}{\partial Y}\right)^2}}\left\{\dfrac{\partial^2\Psi}{\partial X^2}\left[\left(\dfrac{\partial\Psi}{\partial Y}\right)^2 + 2\left(\dfrac{\partial\Psi}{\partial X}\right)^2\right] + \dfrac{\partial^2\Psi}{\partial Y^2}\left[\left(\dfrac{\partial\Psi}{\partial X}\right)^2 + 2\left(\dfrac{\partial\Psi}{\partial Y}\right)^2\right] + 2\dfrac{\partial\Psi}{\partial X}\dfrac{\partial\Psi}{\partial Y}\dfrac{\partial^2\Psi}{\partial X\partial Y}\right\} = 0 \\ \dfrac{\partial^2\Psi}{\partial X^2} + \dfrac{\partial^2\Psi}{\partial Y^2} + \Omega = 0 \end{cases} \tag{70}$$

We discretize the system (70) using central finite differences on an $N \times N$ equidistant grid ($N = 201$) and solve the resulting nonlinear algebraic system using a Gauss-Seidel iteration.

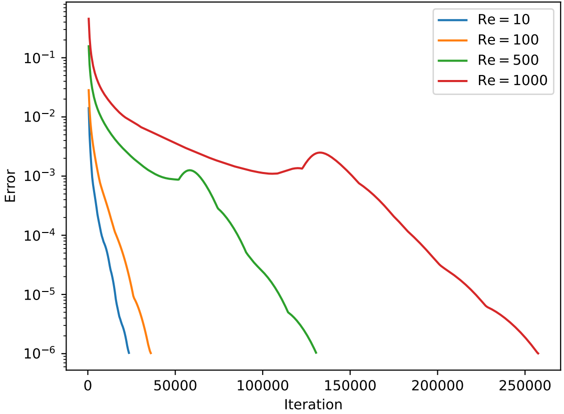
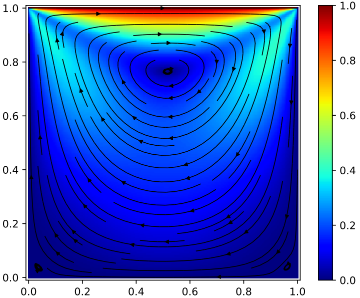
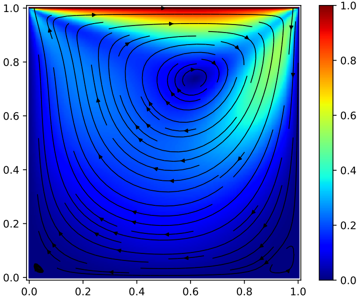


Figure: Convergence of G-S iteration for different Re

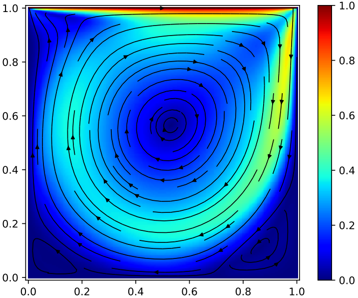
To validate the numerical method, we consider the classical (non-porous) lid-driven cavity flow problem. In this case, the flow is modeled by the system (66) with $\varphi = 1$ and $\mathrm{Da} \to \infty$, i.e., the Navier-Stokes system.



(a) $\mathrm{Re} = 10$



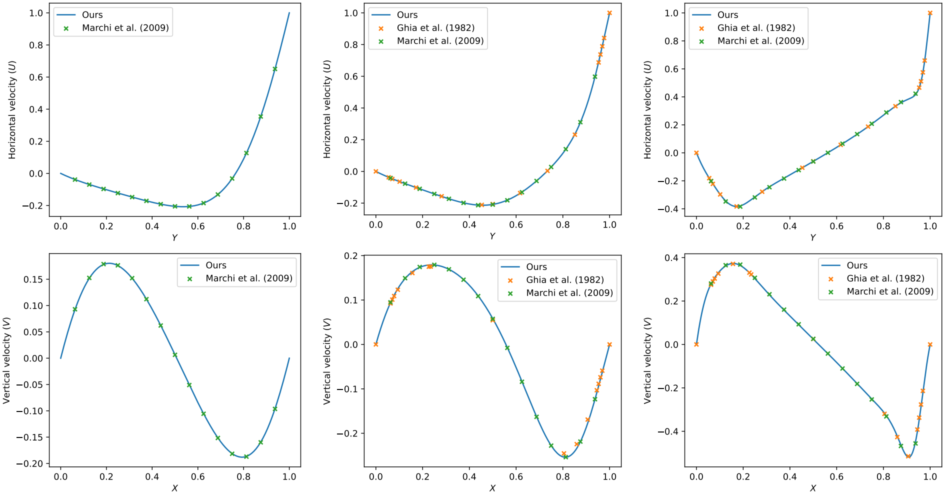
(b) $\mathrm{Re} = 100$



(c) $\mathrm{Re} = 1000$

Figure: Velocity magnitude and streamlines at different Reynolds numbers

We compare the velocity through the geometric center of the cavity with the values provided in[21,22]:

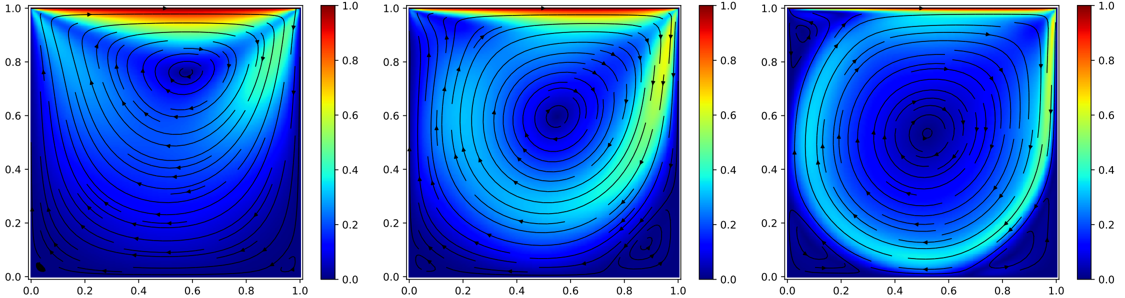


(a) $\mathrm{Re} = 10$ (b) $\mathrm{Re} = 100$ (c) $\mathrm{Re} = 1000$

[21] Ghia, UKNG, Ghia, Kirti N, and Shin, CT. "High-Re solutions for incompressible flow using the Navier-Stokes equations and a multigrid method". 1982.

[22] Marchi, Carlos Henrique, Suero, Roberta, and Araki, Luciano Kiyoshi. "The lid-driven square cavity flow: numerical solution with a 1024 x 1024 grid". 2009.

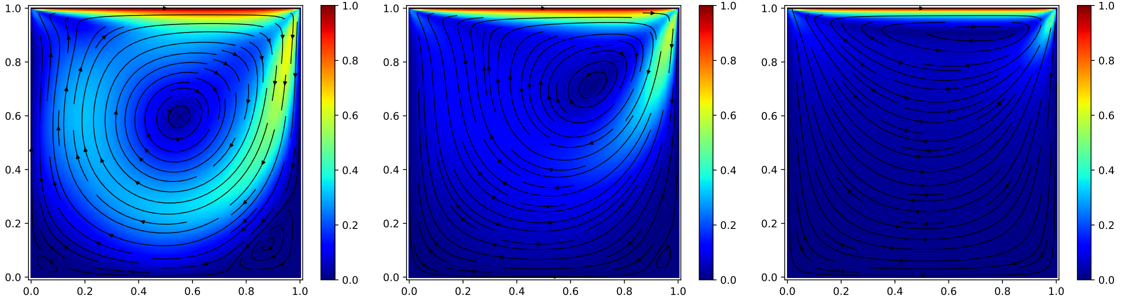
We set $\mathrm{Da} = 0.25, \varphi = 0.2$ and consider $\mathrm{Re} = 10, 100, 1000$:



(a) $\mathrm{Re} = 10$ (b) $\mathrm{Re} = 100$ (c) $\mathrm{Re} = 1000$

Figure: Velocity magnitude and streamlines at different Reynolds numbers

We set $\mathrm{Re} = 100, \varphi = 0.2$ and consider $\mathrm{Da} = 0.25, 0.0025, 0.00025$:



(a) $\mathrm{Da} = 0.25$ (b) $\mathrm{Da} = 0.0025$ (c) $\mathrm{Da} = 0.00025$

Figure: Velocity magnitude and streamlines at different Darcy numbers

The lid-driven cavity flow problem

Bidisperse porous medium case: mathematical model

We now consider the case of a cavity filled with a bidisperse porous medium. The steady state of the flow is governed by the following coupled DFB equations:

$$\begin{cases} \mu\left(\frac{\partial^2 u_f}{\partial x^2}+\frac{\partial^2 u_f}{\partial y^2}\right)-\varphi_f\frac{\partial \pi}{\partial x}-\frac{\mu\varphi_f}{K_f}u_f-\frac{\rho}{\varphi_f}\left(u_f\frac{\partial u_f}{\partial x}+v_f\frac{\partial u_f}{\partial y}\right)-\varphi_f c_f u_f\sqrt{u_f^2+v_f^2}-\varphi_f\zeta(u_f-u_p)=0 \\ \mu\left(\frac{\partial^2 v_f}{\partial x^2}+\frac{\partial^2 v_f}{\partial y^2}\right)-\varphi_f\frac{\partial \pi}{\partial y}-\frac{\mu\varphi_f}{K_f}v_f-\frac{\rho}{\varphi_f}\left(u_f\frac{\partial v_f}{\partial x}+v_f\frac{\partial v_f}{\partial y}\right)-\varphi_f c_f u_f\sqrt{u_f^2+v_f^2}-\varphi_f\zeta(v_f-v_p)=0 \\ \mu\left(\frac{\partial^2 u_p}{\partial x^2}+\frac{\partial^2 u_p}{\partial y^2}\right)-\varphi_p\frac{\partial \pi}{\partial x}-\frac{\mu\varphi_p}{K_p}u_p-\frac{\rho}{\varphi_p}\left(u_p\frac{\partial u_p}{\partial x}+v_p\frac{\partial u_p}{\partial y}\right)-\varphi_p c_p u_p\sqrt{u_p^2+v_p^2}-\varphi_p\zeta(u_p-u_f)=0 \\ \mu\left(\frac{\partial^2 v_p}{\partial x^2}+\frac{\partial^2 v_p}{\partial y^2}\right)-\varphi_p\frac{\partial \pi}{\partial y}-\frac{\mu\varphi_p}{K_p}v_p-\frac{\rho}{\varphi_p}\left(u_p\frac{\partial v_p}{\partial x}+v_p\frac{\partial v_p}{\partial y}\right)-\varphi_f c_p u_p\sqrt{u_p^2+v_p^2}-\varphi_f\zeta(v_p-v_f)=0 \\ \frac{\partial u_f}{\partial x}+\frac{\partial v_f}{\partial y}=0,\ \frac{\partial u_p}{\partial x}+\frac{\partial v_p}{\partial y}=0 \end{cases} \tag{71}$$

where:

where:

- ρ and μ are the density and viscosity of the fluid
- φ_f and K_f are the volume fraction and permeability of the $f-$phase
- φ_p and K_p are the porosity and permeability of the $p-$phase
- ζ is the coefficient for momentum transfer between the two phases (usually taken $\zeta = 1$)

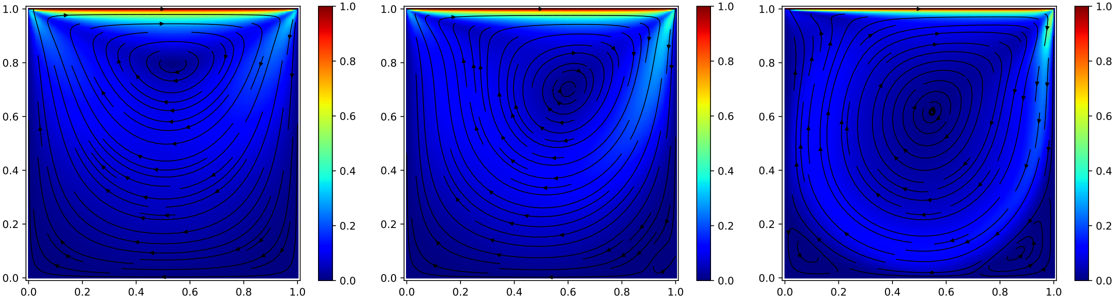
Following Nield and Kuznetsov[23], we introduce the average velocity as:

$$u_{\text{avg}} = \varphi_f u_f + (1 - \varphi_f) u_p, \quad v_{\text{avg}} = \varphi_f v_f + (1 - \varphi_f) v_p \tag{72}$$

We perform similar steps as in the monodisperse case (nondimensionalization, streamfunction–vorticity formulation, discretization using central differences, numerical solution via G-S iteration).

[23] Nield, DA and Kuznetsov, AV. "A note on modeling high speed flow in a bidisperse porous medium". 2013.

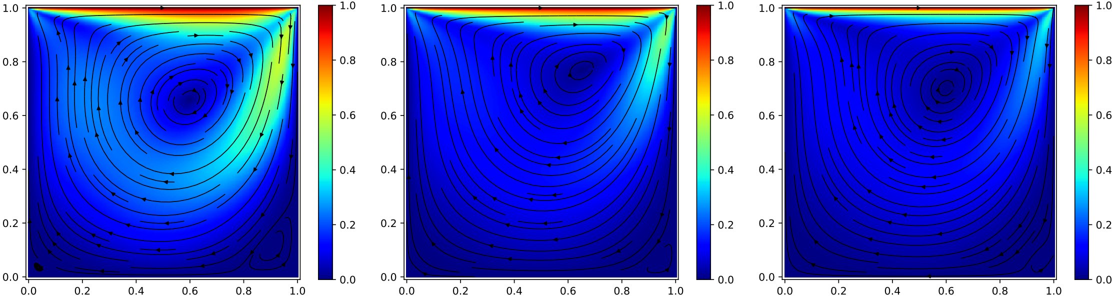
We set $\varphi_f = \varphi_p = 0.4$, $\mathrm{Da}_f = 0.25$, $\mathrm{Da}_p = 0.00025$ and consider $\mathrm{Re} = 10, 100, 1000$:



(a) $\mathrm{Re} = 10$ (b) $\mathrm{Re} = 100$ (c) $\mathrm{Re} = 1000$

Figure: Average velocity magnitude and streamlines at different Reynolds numbers

We set $\varphi_f = \varphi_p = 0.4$, $\mathrm{Da}_f = 0.25$, $\mathrm{Re}_p = 100$ and consider $\mathrm{Da}_p = 0.25, 0.0025, 0.00025$:



(a) $\mathrm{Da}_p = 0.25$ (b) $\mathrm{Da}_p = 0.0025$ (c) $\mathrm{Da}_p = 0.00025$

Figure: Average velocity magnitude and streamlines at different Darcy numbers

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